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Consensus in multi-agent system with communication constraints based on asynchronous dynamics

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Abstract

Consensus of multi-agent system (MAS) in which each agent communicates with the others over wireless network is investigated. For the convenience of calculation, analysis, and discussion, the MAS under the fixed communication topology with communication constraints including random time delay, packet loss and environmental noise is transformed into an asynchronous dynamical system. Thus, the consensus control of MAS is equivalent to the H_∞ control of asynchronous dynamical system. The sufficient conditions of robust exponential stability of MAS with communication constraints are proposed and the corresponding γ-suboptimal H_∞ consensus controllers are designed in terms of linear matrix inequalities (LMIs). The admissible packet loss probability guaranteeing the robust exponential stability of system is also obtained. Subsequently, the corresponding simulation results are provided to demonstrate the effectiveness of the algorithms and the impacts of time delay and packet loss on consensus of system are also analysed in details.

Keywords

Consensus, multi-agent system (MAS), communication constraints, asynchronous dynamics, H_∞

Introduction

Motivated by the applications in cooperative control of unmanned aerial vehicles (Dasgupta et al., 2006), formation control of mobile agents (Oh et al., 2015), distributed data fusion (Gorodetski et al., 2002), sensor networks (La and Sheng, 2013), etc., many researchers have focused their attention on coordination control of multi-agent system (MAS) in recent years. In the environment of MAS cooperative control, it is difficult to realize centralized control due to the causes such as the variations of sensor measurements, the failures of communication channel, and the distribution of information and computing. What is more, It is depended on reliable data transmission, which results in the lack of robustness and real-time. However, the decentralized control is suitable to meet the requirements such as reliability, scalability, agility, quick and easy maintenance, and low cost. And the system performance-real time balance can be achieved in the decentralized control structure (Zhou et al., 2008). Consensus algorithm design is one of the important problems encountered in decentralized control of communicating-agent system. Consensus has been extensively studied and many theoretical and practical issues have been reported in (Jin, 2007; Saboori and Khorasani, 2014; Yang et al., 2016).

In recent years, information is exchanged using a wireless network among agents. The insertion of the communication network in the feed-back control loop makes the analysis and design of the MAS complex. For the distributed consensus control of MAS, time delays may arise naturally, e.g., because of agents moving, the congestion of the communication channels, and the asymmetry of the interactions. Consensus algorithms for MAS with time delay were discussed in (Olfati-Saber and Murray, 2004; Ma et al., 2014; Rong et al., 2012; Zhang et al., 2016). Subha and Liu (2015) discussed the design and practical

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implementation of solving an external consensus problem for MAS. Based on the recursive equation, a prediction strategy was proposed using the transfer function form to overcome the effects of the constant network delay. Xie and Chen (2013) and Jiang, et al. (2015) investigated observer-based consensus control strategies for MAS with time-varying communication delays. Based on the system transformation method, some consensus conditions were established in terms of linear matrix inequality (LMI) in the former. And in the latter, the observer-based consensus protocol was proposed based on the truncated predictor feedback method, algebraic graph theory and Riccati equation. Besides, the leader-following consensus problem of linear MAS with time-varying delays and arbitrary switching topologies was considered in (Chen and Shi, 2015). The MAS with arbitrary switching topologies was formulated as a switched system and then the leader-following consensus problem can be transformed to a stability problem of the switched system. Furthermore, Hu, et al. (2015) studied the consensus problem for MAS in the cooperation-competition network. By combining the Lyapunov theory with the synchronization manifold method, two kinds of time-delayed control schemes were designed in the competition sub-network. In network transmission, the data collision and failure of node competition tend to result in packet loss. Taking into account the fact that packet loss can degrade the performance of MAS and can even lead to an unstable system. In (Wen et al., 2013; Li and Su, 2016), the consensus problems were formulated over networks with random packet loss. (Wen et al., 2013) studied the distributed consensus problem of MAS with occasionally missing control inputs. By appropriately constructing a Lyapunov function and using tools from the M-matrix theory, some sufficient conditions were provided. (Li and Su, 2016) investigated consensus problem for MAS under a fixed strongly connected topology, where each agent can only communicate with its neighbors on some disconnected time intervals. Based on the Lyapunov stability theory and the intermittent control method, some novel and simple criteria were derived for consensus of MAS. Since noise is ubiquitous in both natural and man-made systems, the motion of an agent group was inevitably subjected to noise in the environment (Hu et al., 2014; Li and Zhang, 2010). Jameel et al. (2016) addressed output-feedback-based distributed adaptive consensus control of Lipschitz nonlinear MAS subjected to external disturbances. A robust adaptive fully distributed consensus algorithm was suggested based on the relative outputs of neighboring agents and the adaptive coupling weights, under which consensus was reached between the nonlinear systems for all undirected connected communication topologies in terms of LMIs.

Different kinds of communication constraints were considered simultaneously in the relevant literatures. The consensus results can be obtained in (Almeida et al., 2012; Yan et al., 2014) for MAS with time delay and packet loss. (Almeida et al., 2012) addressed the consensus problem of MAS that evolve in continuous-time and exchange information at discrete-time instants. It was shown that consensus is reached asymptotically by reducing the original problem involving continuous-time variables and asynchronous communications to a discrete-time equivalent and using known results for discrete-time consensus. In (Yan et al., 2014), a consensus algorithm was proposed depending only on periodic sampling and transmitting data for MAS with time delay and packet loss in order to be convenient for practical implementation. The consensus problems of MAS with packet loss and environmental noise simultaneously were discussed in (Yin et al., 2014; Wang et al., 2015). (Yin et al., 2014) studied the consensus control of MAS with Markov random packet loss and environmental noise. The stochastic uncertain factors in the network were transformed into some uncertain parameters of uncertain systems. By analyzing the stability of the whole system and using LMI method, a consistent consensus control algorithm for MAS was designed. In (Wang et al., 2015), packet loss was compensated by the state observer and some necessary and sufficient mean square consensus conditions were obtained. In addition, Cai Y (2013) proposed a particle swarm optimization-based approach combined with a fuzzy obstacle avoidance module for MAS to accomplish consensus in unknown environments. The consensus problem was considered for a team of second-order mobile agents with variable delays, occasional packet losses and environmental noise in (Zhang and Tian, 2010). A queuing mechanism was applied and the switching process of the communication topology of the network was modeled as a

Bernoulli random process. In such a framework, a mean-square consensus condition was proposed by discussing the stability of a reduced-order system and then a condition for the solvability of the mean-square consensus problem was obtained by using the perturbation argument and Routh criterion.

It is worth pointing out that, to the best of the authors' knowledge, few relevant work fully addresses communication constraints for MAS (Zhang and Tian, 2010). Motivated by the above analysis, this paper, a continuation and improvement of the previous work (Yin et al., 2014), makes further endeavors to develop a more realistic consensus algorithm which fully addresses communication constraints including random time delay, packet loss and environmental noise for MAS. For the convenience of analysis, the MAS with communication constraints was divided into two cases: (A) the system is subjected to the random short time delay, packet loss and environmental noise; (B) the system is subjected to the random long time delay, packet loss and environmental noise. The main contributions of this paper can be summarized as: the first one is that the MAS with communication constraints can be transformed into an asynchronous dynamical system; thus, the MAS consensus to be solved is simplified as analyzing the robust exponential stability of an asynchronous dynamical system. The second one is that sufficient stability criteria for asynchronous dynamical system are derived, which ensures that the consensus in MAS can be achieved. Based on the criteria, the consensus γ-suboptimal H_∞ controller gains are designed by constructing the Lyapunov functions and using the LMI method. The third one is that the admissible packet loss probability guaranteeing the robust exponential stability of the system is also obtained.

The outline of this paper is as follows. The preliminaries and problem formulation are introduced in the second section. The models of MAS with communication constraints including random time delay, packet loss and environmental noise are formulated in the third section. Corresponding consensus analyses and designs for systems with communication constraints are given in the fourth section. The simulation results are provided in the fifth section. Finally, the conclusion is drawn in the sixth section.

Preliminaries and problem formulation

Firstly, the communication topology of MAS is usually modeled by the directed digraph $G=\left(\mathbf{V},\mathbf{E},\boldsymbol{A}_l\right)$. $\mathbf{V}=\left\{v_1,\cdots,v_n\right\}$ denotes the node set in which node v_i represents agent i. $\mathbf{E}\subseteq\mathbf{V}\times\mathbf{V}$ denotes the edge set whose elements denote the directed communication links between agents. A directed edge $e_{ij}=\left(v_i,v_j\right)$ means that there is a communication link from agent j to agent i. If edge $e_{ij}\in\mathbf{E}$, then v_j is one of the neighbors of v_i. The set of neighbors of v_i is denoted by $\mathbf{N}_i=\left\{v_j\in\mathbf{V}\middle|e_{ij}\in\mathbf{E}\right\}$. The graph is called directed complete graph, if every two nodes are connected through two sides in the opposite direction. $\boldsymbol{A}_l=\left[a_{ij}\right]\in\mathbf{R}^{n\times n}$ denotes the adjacency matrix whose elements can be defined as $a_{ij}=1\Leftrightarrow\left(v_i,v_j\right)\in\mathbf{E}$ and $a_{ij}=0\Leftrightarrow\left(v_i,v_j\right)\notin\mathbf{E}$. It is assumed that there is no self-loop in G, i.e., $e_{ii}\notin\mathbf{E}$, $a_{ii}=0$ for all $i=1,\cdots,n$. The Laplacian matrix of a directed graph G is defined as $\boldsymbol{L}=\left[l_{ij}\right]\in\mathbf{R}^{n\times n}$, and $l_{ij}=\begin{cases}-a_{ij}, & i\neq j\\ \sum_{j=1}^{n}a_{ij}, & i=j\end{cases}$ $i,j=1,\cdots,n$.

A directed graph with four nodes is shown in Figure 1. It is clear that this directed graph is complete graph. The associated adjacency matrix $\boldsymbol{A}_l$, and Laplacian matrix $\boldsymbol{L}$ are listed below.

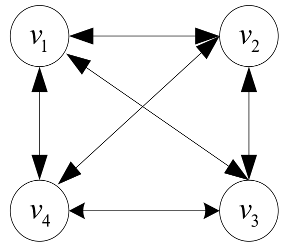


Figure 1. An example: directed graph with four nodes

$$A_l=\begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix},\quad L=\begin{bmatrix}3&-1&-1&-1\\-1&3&-1&-1\\-1&-1&3&-1\\-1&-1&-1&3\end{bmatrix}.$$

Secondly, consider a system consisting of n identical agents with continuous-time linear time-invariant (LTI) dynamics subject to the environmental noise, the model of each agent can be described by (Wang and Ding, 2016)

$$\begin{cases}\dot{\boldsymbol{x}}_i(t)=\boldsymbol{A}\boldsymbol{x}_i(t)+\boldsymbol{B}\boldsymbol{u}_i(t)+\boldsymbol{G}\xi_i(t)\\ \boldsymbol{y}_i(t)=\boldsymbol{C}\boldsymbol{x}_i(t)\end{cases}\tag{1}$$

where $i=1,2,\cdots,n$ is the label of each agent. $\boldsymbol{x}_i(t)\in\mathbf{R}^n$ is the state vector of the agent i, $\boldsymbol{u}_i(t)\in\mathbf{R}^{m\times n}$ is the control input, $\xi_i(t)$ is assumed to be white noise which belongs to $L_2^m[0,\infty)$, and $\boldsymbol{y}_i(t)\in\mathbf{R}^m$ is the output vector of the agent i. $\boldsymbol{A}\in\mathbf{R}^{n\times n}$, $\boldsymbol{B}\in\mathbf{R}^{n\times m}$, $\boldsymbol{C}\in\mathbf{R}^{n\times n}$ and $\boldsymbol{G}\in\mathbf{R}^{n\times m}$ are constant matrices with appropriate dimensions. It is assumed that $(\boldsymbol{A},\boldsymbol{B})$ is stabilizable, $(\boldsymbol{A},\boldsymbol{C})$ is detectable. Without loss of generality, $\boldsymbol{B}$ is of full column rank.

The consensus algorithm for the i th agent is taken as：

$$\boldsymbol{u}_i(t)=\boldsymbol{K}\sum_{j=1,j\in N_i}^{n}a_{ij}\left(\boldsymbol{x}_j(t)-\boldsymbol{x}_i(t)\right)\tag{2}$$

where $\boldsymbol{K}\in\mathbf{R}^{q\times p}$ is the control gain. $a_{ij}\in\{0,1\}$ denotes the wireless communication connection from j to i. $a_{ij}=1$, if the connection is good, otherwise, $a_{ij}=0$.

The above consensus algorithm (2) requires that agent i can obtain information from its neighbors in N_i timely and accurately, that is, it assumes zero communication time-delay and accurate information exchange among agents. When the communication time delay of the state information transmitted from agent j to agent i is $\tau(t)$, so the following delayed stochastic consensus algorithm (3) is considered

$$\boldsymbol{u}_i(t)=\boldsymbol{K}\sum_{j=1,j\in N_i}^{n}a_{ij}\left(\boldsymbol{x}_j\left(t-\tau(t)\right)-\boldsymbol{x}_i\left(t-\tau(t)\right)\right)\tag{3}$$

Let $\boldsymbol{z}_i(t)=\boldsymbol{x}_i(t)-\boldsymbol{x}_1(t)$ and $\boldsymbol{w}_i(t)=\xi_i(t)-\xi_1(t)$ $\left(i=2,3,\cdots,n\right)$. If the control network is introduced into system (1), under a delayed consensus algorithm (3), Equation (4) can be obtained

$$
\begin{cases}
\dot{z}_i(t) = \dot{x}_i(t) - \dot{x}_1(t) \\
\quad = Ax_i(t) + Bu_i(t) + G\xi_i(t) - \left[Ax_1(t) + Bu_1(t) + G\xi_1(t)\right] \\
\quad = \left[Ax_i(t) - Ax_1(t)\right] + BK\sum_{j=1, j\in N_i}^{n} a_{ij}\left(x_j\left(t-\tau(t)\right) - x_i\left(t-\tau(t)\right)\right) - BK\sum_{j=1, j\in N_i}^{n} a_{1j}\left(x_j\left(t-\tau(t)\right) - x_1\left(t-\tau(t)\right)\right) + \left[G\xi_i(t) - G\xi_1(t)\right] \\
\quad = Az_i(t) + BK\sum_{j=1, j\in N_i}^{n}(a_{ij} - a_{1j})x_j\left(t-\tau(t)\right) - BK\sum_{j=1, j\in N_i}^{n} a_{ij}x_i\left(t-\tau(t)\right) + BK\sum_{j=1, j\in N_i}^{n} a_{1j}x_1\left(t-\tau(t)\right) \\
\qquad + BK\sum_{j=1, j\in N_i}^{n} a_{ij}x_1\left(t-\tau(t)\right) - BK\sum_{j=1, j\in N_i}^{n} a_{ij}x_1\left(t-\tau(t)\right) + Gw_i(t) \\
\quad = Az_i(t) + BK\sum_{j=2,\ j\in N_i}^{n}(a_{ij} - a_{1j})z_j\left(t-\tau(t)\right) - BK\sum_{j=1,\ j\in N_i}^{n} a_{ij}z_i\left(t-\tau(t)\right) + Gw_i(t) \\
\bar{y}_i(t) = C\left(x_i(t) - x_1(t)\right) = Cz_i(t)
\end{cases}
\tag{4}
$$

Converting Equation (4) and substituting $a_{ii} = 0$, it can be obtained as follows:

$$
\begin{bmatrix} z_2(t) \\ z_3(t) \\ z_4(t) \\ \vdots \\ z_n(t) \end{bmatrix} = \begin{bmatrix} A & 0 & 0 & \cdots & 0 \\ 0 & A & 0 & \cdots & 0 \\ 0 & 0 & A & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & A \end{bmatrix} \begin{bmatrix} z_2(t) \\ z_3(t) \\ z_4(t) \\ \vdots \\ z_n(t) \end{bmatrix} - \left(\tilde{L} \otimes BK\right) \begin{bmatrix} z_2(t-\tau) \\ z_3(t-\tau) \\ z_4(t-\tau) \\ \vdots \\ z_n(t-\tau) \end{bmatrix} + \begin{bmatrix} G & 0 & 0 & \cdots & 0 \\ 0 & G & 0 & \cdots & 0 \\ 0 & 0 & G & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & G \end{bmatrix} \begin{bmatrix} w_2(t) \\ w_3(t) \\ w_4(t) \\ \vdots \\ w_n(t) \end{bmatrix} \tag{4-1}
$$

where $\tilde{L} = \begin{bmatrix} a_{12} + \sum_{j=1,\ j\in N_i}^{n} a_{2j} & -a_{23} - (-a_{13}) & -a_{24} - (-a_{14}) & \cdots & -a_{2n} - (-a_{1n}) \\ -a_{32} - (-a_{12}) & a_{13} + \sum_{j=1,\ j\in N_i}^{n} a_{3j} & -a_{34} - (-a_{14}) & \cdots & -a_{3n} - (-a_{1n}) \\ -a_{42} - (-a_{12}) & -a_{43} - (-a_{13}) & a_{14} + \sum_{j=1,\ j\in N_i}^{n} a_{4j} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -a_{(n-1)n} - (-a_{1n}) \\ -a_{n2} - (-a_{12}) & -a_{n3} - (-a_{13}) & \cdots & -a_{n(n-1)} - (-a_{1(n-1)}) & a_{1n} + \sum_{j=1,\ j\in N_i}^{n} a_{nj} \end{bmatrix}$.

From the equation (4-1), it can be seen that $\tilde{L} = \left[\tilde{l}_{ij}\right] \in \mathbf{R}^{(n-1)\times(n-1)}$, the elements of $\tilde{L}$ can be determined as $\tilde{l}_{ij} = a_{1(j+1)} + \sum_{j=1}^{n} a_{(i+1)(j+1)}$ if $i = j$; $\tilde{l}_{ij} = -a_{(i+1)(j+1)} - (-a_{1(j+1)})$, if $i \neq j$.

$$
\begin{bmatrix} \bar{y}_2(t) \\ \bar{y}_3(t) \\ \bar{y}_4(t) \\ \vdots \\ \bar{y}_n(t) \end{bmatrix} = \begin{bmatrix} C & 0 & 0 & \cdots & 0 \\ 0 & C & 0 & \cdots & 0 \\ 0 & 0 & C & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & C \end{bmatrix} \begin{bmatrix} z_2(t) \\ z_3(t) \\ z_4(t) \\ \vdots \\ z_n(t) \end{bmatrix} \tag{4-2}
$$

Let $z = [z_2^{\mathrm{T}}, z_3^{\mathrm{T}}, \cdots, z_n^{\mathrm{T}}]^{\mathrm{T}}$, $w = [w_2^{\mathrm{T}}, w_3^{\mathrm{T}}, \cdots, w_n^{\mathrm{T}}]^{\mathrm{T}}$, $\bar{y} = [\bar{y}_2^{\mathrm{T}}, \bar{y}_3^{\mathrm{T}}, \cdots, \bar{y}_n^{\mathrm{T}}]^{\mathrm{T}}$, then (5) can be obtained

$$
\begin{cases}
\dot{z}(t) = \left(I_{n-1} \otimes A\right)z(t) - \left(\tilde{L} \otimes BK\right)z\left(t-\tau(t)\right) + \left(I_{n-1} \otimes G\right)w(t) \\
\bar{y}(t) = \left(I_{n-1} \otimes C\right)z(t)
\end{cases}
\tag{5}
$$

where $\tilde{\boldsymbol{L}}_{ij} = \left[\tilde{l}_{ij}\right]_{(n-1)\times(n-1)}$, $\tilde{l}_{ij} = l_{(i+1)(j+1)} - l_{1(j+1)}$, $l_{ij} = \begin{cases} -a_{ij}, & i \neq j \\ \sum_{j=1}^{n} a_{ij}, & i = j \end{cases}$; "$\otimes$" denotes the Kronecker product.

Definition 1. Given a positive scalar γ, the transformed MAS (5) is said to achieve consensus with a guaranteed H_∞ performance $\gamma > 0$ if and only if the following two requirements hold:

(1) The MAS (5) with $\boldsymbol{w}(t) \equiv 0$ can reach consensus. That is, under control algorithms, $\lim_{t\to\infty} \boldsymbol{z}(t) = 0$ holds, i.e. $\lim_{t\to\infty} \boldsymbol{z}_i(t) = \lim_{t\to\infty}\left(\boldsymbol{x}_i(t) - \boldsymbol{x}_1(t)\right) = 0 \ (i = 2,3,\cdots,n)$.

(2) Under the zero-initial condition, the MAS (7) satisfies the dissipation inequality $\int_0^\infty \left\|\bar{\boldsymbol{y}}(t)\right\|^2 dt < \gamma^2 \int_0^\infty \left\|\boldsymbol{w}(t)\right\|^2 dt$ for all $\boldsymbol{w}(t) \neq 0$ and $\forall \boldsymbol{w}(t) \in L_2[0,\infty)$; or equivalently, the H_∞ performance index satisfies $J = \int_0^\infty \left[\bar{\boldsymbol{y}}^{\mathrm{T}}(t)\bar{\boldsymbol{y}}(t) - \gamma^2 \boldsymbol{w}^{\mathrm{T}}(t)\boldsymbol{w}(t)\right] dt < 0$.

The MAS achieves consensus under fixed communication topology, if and only if the control gain $\boldsymbol{K}$ simultaneously stabilizes $N-1$ subsystems like (Zhang and Tian, 2012)

$$\begin{cases} \dot{\boldsymbol{z}}_i(t) = \boldsymbol{A}\boldsymbol{z}_i(t) - \lambda_i \boldsymbol{B}\boldsymbol{K}\boldsymbol{z}_i\left(t-\tau(t)\right) + \boldsymbol{G}\boldsymbol{w}_i(t) \\ \bar{\boldsymbol{y}}_i(t) = \boldsymbol{C}\boldsymbol{z}_i(t) \end{cases} \tag{6}$$

where $\lambda_i \ \left(i = 2,3,\cdots,n\right)$ are the eigenvalues of $\tilde{\boldsymbol{L}}$.

As the fixed communication topology of the MAS considered in this paper is a complete graph. Its adjacency matrix $\boldsymbol{A}_l$ is symmetric with the specific that the diagonal elements are 0 and the others are 1, so the non-zero eigenvalues of $\tilde{\boldsymbol{L}}$ are equal. Let $\lambda = \lambda_i \ \left(i = 2,3,\cdots,n\right)$. Because all the subsystems are identical, let $\boldsymbol{A}_m = \boldsymbol{A}$, $\boldsymbol{B}_m = -\lambda\boldsymbol{B}$, then (6) can be rewritten as

$$\begin{cases} \dot{\boldsymbol{z}}_i(t) = \boldsymbol{A}_m\boldsymbol{z}_i(t) + \boldsymbol{B}_m\boldsymbol{u}_i(t) + \boldsymbol{G}\boldsymbol{w}_i(t) \\ \boldsymbol{u}_i(t) = \boldsymbol{K}\boldsymbol{z}_i\left(t-\tau(t)\right) \qquad\qquad (i = 2,3,\cdots,n) \\ \bar{\boldsymbol{y}}_i(t) = \boldsymbol{C}\boldsymbol{z}_i(t) \end{cases} \tag{7}$$

Remark 1. The consensus control problem considered in this paper is to design a stable controller assuring robust exponential stability and a prescribed H_∞ performance level for the system, using only the relative state information to ensure that all the subsystems converge to the identical state, which means that the MAS reaches a consensus. The MAS with communication constraints is divided into two cases: one case is that the system is subjected to random short time delay, packet loss, and environmental noise, it is denoted as case A; the other case is that the system is subjected to random long time delay, packet loss, and environmental noise, it is denoted as case B.

Modeling for MAS with communication constraints

In this section, the models of MAS with communication constraints in two cases are established.

For the MAS (7), in network transmission, the data collision and failure of node competition tend to result in packet loss. If agent i has not received the information of the neighbor agent at time k, the control signal at time $k-1$ should be adopted as the control signal during this period, i.e. $\boldsymbol{u}_i(k) = \boldsymbol{u}_i(k-1)$. The network with packet loss can be treated as a switch with certain rate. The data transmission is successful, when the switch is closed, it is denoted as event S_1 and the rate of occurrence is set as

η ($0 < \eta \le 1$); The packet loss happens, when the switch is open, it is denoted as event S_2, whose rate of occurrence is $1-\eta$. Hence the MAS can be equivalent to an asynchronous dynamical system. The simplified asynchronous dynamical system model is shown as Figure 2.

Remark 2. Roughly speaking, asynchronous dynamical systems are systems that incorporate both discrete and continuous dynamics, with the discrete dynamics governed by finite automata, and the continuous dynamics represented by ordinary differential (or difference) equations at each discrete state. The discrete dynamics is driven asynchronously by discrete events, which are assumed to occur at a fixed rate.

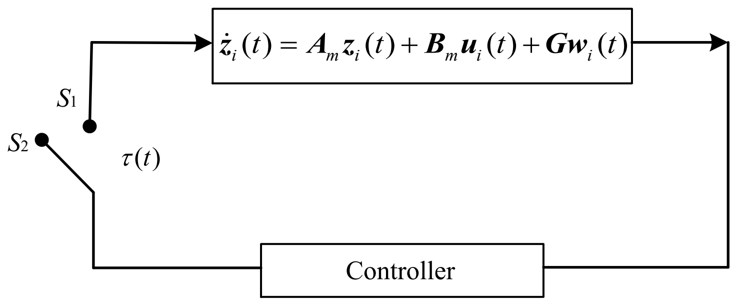


Figure 2. The simplified asynchronous dynamical system model

Modeling for MAS in case A

Suppose the sampling period in the MAS is T, and the random short time delay satisfies $0 < \tau_k \le T$. The MAS model can be established as follows.

The MAS (7) can be transformed into a discrete LTI system with matrix theory, due to the existence of short time delay $\tau(t)$ in (7). From the matrix theory, matrix $\boldsymbol{A}_m$ is diagonalizable if and only if the sum of the dimensions of the eigenspaces is n, or, equivalently, if and only if $\boldsymbol{A}_m$ has linearly independent eigenvectors. $\boldsymbol{A}_m = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1}$, if .. is diagonalizable, i.e. $\boldsymbol{\Lambda} = \boldsymbol{S}^{-1}\boldsymbol{A}_m\boldsymbol{S}$, where $\boldsymbol{\Lambda}=diag\left(\lambda_{m1}, \lambda_{m2}, \cdots, \lambda_{mn}\right)$, and $\lambda_{m1}, \lambda_{m2}, \cdots \lambda_{mn}$ are the eigenvalues of $\boldsymbol{A}_m$. Without loss of generality, suppose $\lambda_{m1}, \lambda_{m2}, \cdots, \lambda_{mn}$ are all nonzero, then $\boldsymbol{B}_0 = \boldsymbol{S}diag\left(-\frac{1}{\lambda_{m1}}, \cdots, -\frac{1}{\lambda_{mn}}\right)\boldsymbol{S}^{-1}\boldsymbol{B}_m$, $\boldsymbol{B}_1 = \boldsymbol{S}diag\left(\frac{1}{\lambda_{m1}}e^{\lambda_{m1}T}, \cdots, \frac{1}{\lambda_{mn}}e^{\lambda_{mn}T}\right)\boldsymbol{S}^{-1}\boldsymbol{B}_m$, $\boldsymbol{D} = \boldsymbol{S}diag\left(\frac{1}{\lambda_{m1}}e^{\lambda_{m1}a_1}, \cdots, \frac{1}{\lambda_{mn}}e^{\lambda_{mn}a_n}\right)$, $\boldsymbol{F}\left(\tau_k\right) = diag\left(e^{\lambda_{m1}(T-\tau_k-a_1)}, \cdots, e^{\lambda_{mn}(T-\tau_k-a_n)}\right)$, $\boldsymbol{E} = \boldsymbol{S}^{-1}\boldsymbol{B}_m$, where constants $a_1, \cdots, a_n$ should ensure $e^{\lambda_{mi}(T-\tau_k-a_i)} < 1, i = 1, 2, \cdots n$ to satisfy $\boldsymbol{F}^{\mathrm{T}}(\tau_k)\boldsymbol{F}(\tau_k) \le \boldsymbol{I}$ (Zhang and Qiu, 2007).

According to the above deduction, when event S_1 occurs, the discrete model of the system is represented by:

$$\begin{cases} \boldsymbol{z}_i(k+1) = \boldsymbol{A}_k \boldsymbol{z}_i(k) + \left(\boldsymbol{B}_0 + \boldsymbol{D}\boldsymbol{F}(\tau_k)\boldsymbol{E}\right)\boldsymbol{u}_i(k) \\ \qquad + \left(\boldsymbol{B}_1 - \boldsymbol{D}\boldsymbol{F}(\tau_k)\boldsymbol{E}\right)\boldsymbol{u}_i(k-1) + \boldsymbol{G}_w \boldsymbol{w}_i(k) \\ \overline{\boldsymbol{y}}_i(k) = \boldsymbol{C}\boldsymbol{z}_i(k) \end{cases} \tag{8}$$

where $\boldsymbol{A}_k = e^{\boldsymbol{A}_m T}$, $\boldsymbol{G}_w = \int_0^T e^{\boldsymbol{A}_m s} ds \boldsymbol{G}$, $\boldsymbol{B}_0$, $\boldsymbol{B}_1$, $\boldsymbol{D}$ and $\boldsymbol{E}$ are constant matrices. τ_k is the discretization form of $\tau(t)$. $\boldsymbol{F}(\tau_k)$ is subject to τ_k, and it satisfies $\boldsymbol{F}^{\mathrm{T}}(\tau_k)\boldsymbol{F}(\tau_k) \le \boldsymbol{I}$.

When event S_2 occurs, the discrete model of the system is represented as

$$\begin{cases} \boldsymbol{z}_i(k+1) = \boldsymbol{A}_k \boldsymbol{z}_i(k) + \left(\boldsymbol{B}_0 + \boldsymbol{B}_1\right)\boldsymbol{u}_i(k-1) + \boldsymbol{G}_w \boldsymbol{w}_i(k) \\ \bar{\boldsymbol{y}}_i(k) = \boldsymbol{C}\boldsymbol{z}_i(k) \end{cases} \tag{9}$$

Define $\tilde{\boldsymbol{z}}_i(k) = [\boldsymbol{z}_i^{\mathrm{T}}(k) \quad \boldsymbol{z}_i^{\mathrm{T}}(k-1)]^{\mathrm{T}}$ as the augmented state vector, and $\boldsymbol{u}_i(k) = \boldsymbol{K}\boldsymbol{z}_i(k)$ as the control law.

When event S_1 occurs, the augmented closed-loop system for (8) is

$$\begin{cases} \tilde{\boldsymbol{z}}_i(k+1) = \boldsymbol{\Psi}_1 \tilde{\boldsymbol{z}}_i(k) + \tilde{\boldsymbol{G}}\boldsymbol{w}_i(k) \\ \tilde{\boldsymbol{y}}_i(k) = \tilde{\boldsymbol{C}}\tilde{\boldsymbol{z}}_i(k) \end{cases} \tag{10}$$

where $\boldsymbol{\Psi}_1 = \begin{bmatrix} \boldsymbol{A}_k + \left(\boldsymbol{B}_0 + \boldsymbol{D}\boldsymbol{F}(\tau_k)\boldsymbol{E}\right)\boldsymbol{K} & \left(\boldsymbol{B}_1 - \boldsymbol{D}\boldsymbol{F}(\tau_k)\boldsymbol{E}\right)\boldsymbol{K} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}, \tilde{\boldsymbol{G}} = \begin{bmatrix} \boldsymbol{G}_w \\ \boldsymbol{0} \end{bmatrix}, \ \tilde{\boldsymbol{C}} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{0} \end{bmatrix}$.

When event S_2 occurs, the augmented closed-loop system for (9) is

$$\begin{cases} \tilde{\boldsymbol{z}}_i(k+1) = \boldsymbol{\Psi}_2 \tilde{\boldsymbol{z}}_i(k) + \tilde{\boldsymbol{G}}\boldsymbol{w}_i(k) \\ \tilde{\boldsymbol{y}}_i(k) = \tilde{\boldsymbol{C}}\tilde{\boldsymbol{z}}_i(k) \end{cases} \tag{11}$$

where $\boldsymbol{\Psi}_2 = \begin{bmatrix} \boldsymbol{A}_k & \left(\boldsymbol{B}_0 + \boldsymbol{B}_1\right)\boldsymbol{K} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}, \ \tilde{\boldsymbol{G}} = \begin{bmatrix} \boldsymbol{G}_w \\ \boldsymbol{0} \end{bmatrix}, \ \tilde{\boldsymbol{C}} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{0} \end{bmatrix}$.

So the MAS with random short time delay, packet loss, and environmental noise can be described as an asynchronous dynamical system with environmental noise, i.e. $\tilde{\boldsymbol{z}}_i(k+1) = \boldsymbol{\Psi}_s \tilde{\boldsymbol{z}}_i(k) + \tilde{\boldsymbol{G}}\boldsymbol{w}_i(k), \ (s = 1, 2)$.

Modeling for MAS in case B

When the delays can be longer than one sampling period (say, $0 < \tau_k \le mT$, $m > 1$, $m \in \mathbf{N}^+$), the controlled plant may receive zero, one, or even more than one (up to m+1) control signal(s) in a sampling period. The case that control signal(s) received in a sampling period may be zero or more than one is complicated, this will be discussed in detail in another paper. In the special case where $(m-1)T < \tau_k < mT$ for all k, one control signal is received in every sampling period for $k > m$. In this case, the analysis is as follows.

Let $\tau_k' = \tau_k - (m-1)T$, similar to the derivation of (8), when event S_1 occurs, the discrete model of the system is represented as

$$\begin{cases} \boldsymbol{z}_i(k+1) = \boldsymbol{A}_k \boldsymbol{z}_i(k) + \Big(\boldsymbol{B}_0 + \boldsymbol{D}\boldsymbol{F}(\tau_k')\boldsymbol{E}\Big)\boldsymbol{u}_i\left(k-(m-1)\right) \\ \qquad\qquad + \Big(\boldsymbol{B}_1 - \boldsymbol{D}\boldsymbol{F}(\tau_k')\boldsymbol{E}\Big)\boldsymbol{u}_i(k-m) + \boldsymbol{G}_w \boldsymbol{w}_i(k) \\ \bar{\boldsymbol{y}}_i(k) = \boldsymbol{C}\boldsymbol{z}_i(k) \end{cases} \tag{12}$$

where $\boldsymbol{A}_k = e^{A_m T}$, $\boldsymbol{G}_w = \int_0^T e^{A_m s} ds \boldsymbol{G}$, $\boldsymbol{B}_0$, $\boldsymbol{B}_1$, $\boldsymbol{D}$ and $\boldsymbol{E}$ are constant matrices. $\boldsymbol{F}(\tau_k')$ is subject to τ_k', and it satisfies $\boldsymbol{F}^{\mathrm{T}}(\tau_k')\boldsymbol{F}(\tau_k') \le \boldsymbol{I}$.

When event S_2 occurs, the discrete model of the system is represented as

$$\begin{cases} \boldsymbol{z}_i(k+1) = \boldsymbol{A}_k \boldsymbol{z}_i(k) + \boldsymbol{B}_k \boldsymbol{u}_i(k-1) + \boldsymbol{G}_w \boldsymbol{w}_i(k) \\ \overline{\boldsymbol{y}}_i(k) = \boldsymbol{C}\boldsymbol{z}_i(k) \end{cases} \tag{13}$$

where $\boldsymbol{B}_k = \int_0^T e^{A_m s} ds \boldsymbol{B}_m$.

Define $\hat{\boldsymbol{z}}_i(k) = \begin{bmatrix} \boldsymbol{z}_i^{\mathrm{T}}(k) & \boldsymbol{u}_i^{\mathrm{T}}(k-1) & \boldsymbol{u}_i^{\mathrm{T}}(k-2) & \cdots & \boldsymbol{u}_i^{\mathrm{T}}(k-m) \end{bmatrix}^{\mathrm{T}}$ as the augmented state vector, and $\boldsymbol{u}_i(k) = \boldsymbol{K}\hat{\boldsymbol{z}}_i(k)$ as the control law.

When event S_1 occurs, the augmented closed-loop system for (12) is

$$\begin{cases} \hat{\boldsymbol{z}}_i(k+1) = \left(\boldsymbol{\Sigma} + \boldsymbol{\Theta}\boldsymbol{K}\right)\hat{\boldsymbol{z}}_i(k) + \hat{\boldsymbol{G}}\boldsymbol{w}(k) = \boldsymbol{\Psi}_1 \hat{\boldsymbol{z}}_i(k) + \hat{\boldsymbol{G}}\boldsymbol{w}_i(k) \\ \hat{\boldsymbol{y}}_i(k) = \hat{\boldsymbol{C}}\hat{\boldsymbol{z}}_i(k) \end{cases} \tag{14}$$

where $\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{A}_k & \boldsymbol{0} & \cdots & \boldsymbol{B}_0 + \boldsymbol{D}\boldsymbol{F}(\tau_k{}')\boldsymbol{E} & \boldsymbol{B}_1 - \boldsymbol{D}\boldsymbol{F}(\tau_k{}')\boldsymbol{E} \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}$, $\boldsymbol{\Theta} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{bmatrix}$, $\hat{\boldsymbol{G}} = \begin{bmatrix} \boldsymbol{G}_w \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{bmatrix}$, $\hat{\boldsymbol{C}} = \left[\boldsymbol{C}, \boldsymbol{0}, \boldsymbol{0}, \cdots, \boldsymbol{0}\right]$.

Let $\boldsymbol{H} = \begin{bmatrix} \boldsymbol{A}_k & \boldsymbol{0} & \cdots & \boldsymbol{B}_0 & \boldsymbol{B}_1 \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}$, $\hat{\boldsymbol{D}} = \begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{bmatrix}$, $\hat{\boldsymbol{E}} = \begin{bmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{E} & -\boldsymbol{E} \end{bmatrix}$, then $\boldsymbol{\Psi}_1 = \boldsymbol{H} + \hat{\boldsymbol{D}}\boldsymbol{F}(\tau_k{}')\hat{\boldsymbol{E}} + \boldsymbol{\Theta}\boldsymbol{K}$.

When event S_2 occurs, the augmented closed-loop system for (13) is

$$\begin{cases} \hat{\boldsymbol{z}}_i(k+1) = \boldsymbol{\Psi}_2 \hat{\boldsymbol{z}}_i(k) + \hat{\boldsymbol{G}}\boldsymbol{w}_i(k) \\ \hat{\boldsymbol{y}}_i(k) = \hat{\boldsymbol{C}}\hat{\boldsymbol{z}}_i(k) \end{cases} \tag{15}$$

where $\boldsymbol{\Psi}_2 = \begin{bmatrix} \boldsymbol{A}_k & \boldsymbol{B}_k & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & & \boldsymbol{0} \\ \vdots & \vdots & & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{I} \end{bmatrix}$, $\hat{\boldsymbol{G}} = \begin{bmatrix} \boldsymbol{G}_w \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{bmatrix}$, $\hat{\boldsymbol{C}} = \left[\boldsymbol{C}, \boldsymbol{0}, \boldsymbol{0}, \cdots, \boldsymbol{0}\right]$.

So the MAS with random long time delay, packet loss, and environmental noise can be described as an asynchronous dynamical system with environmental noise, i.e. $\hat{\boldsymbol{z}}_i(k+1) = \boldsymbol{\Psi}_s \hat{\boldsymbol{z}}_i(k) + \hat{\boldsymbol{G}}\boldsymbol{w}_i(k),\ (s = 1, 2)$.

In conclusion, the MAS with communication constraints can be modeled as an asynchronous dynamical system. Then the MAS consensus to be solved is simplified as analyzing the robust exponential stability of an asynchronous dynamical system.

Consensus analysis and design for MAS with communication constraints

The consensus analysis and design of MAS with communication constraints can be simplified as the stability analysis and controller design of the asynchronous dynamical system. Moreover, the packet loss probability of network is vital to stability of

MAS modeled as asynchronous dynamical system. In this section, in the light of the system models in two cases, the corresponding consensus analyses and designs for MAS are given. The admissible packet loss probability guaranteeing the robust exponential stability of the system is also calculated.

Next, we will recall some important Lemmas to be used in the sequel (Zhang and Qiu, 2007).

Lemma 1. For given matrices $\boldsymbol{W}=\boldsymbol{W}^{\mathrm{T}}$, $\boldsymbol{M}$, $\boldsymbol{N}$ and $\boldsymbol{F}$ with appropriate dimensions, $\boldsymbol{W}+\boldsymbol{N}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{M}^{\mathrm{T}}+\boldsymbol{MFN}<0$ holds for all $\boldsymbol{F}$ satisfying $\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\le\boldsymbol{I}$ if and only if there exists $\varepsilon>0$ such that $\boldsymbol{W}+\varepsilon^{-1}\boldsymbol{N}^{\mathrm{T}}\boldsymbol{N}+\varepsilon\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}}<0$.

Lemma 2. Consider an asynchronous dynamical system with rate constraints on events, the system is given by the discretized difference equation, $x(k+1)=f_s\left(x(k)\right)$ $(s=1,2,\cdots,N)$. It is exponentially stable, if there exists a Lyapunov function $V\left(x(k)\right)$: $\mathbf{R}^n\to\mathbf{R}^+$, scalars $\alpha_s>0$ $(s=1,2,\cdots,N)$ and constant $\alpha>1$ such that the following inequalities hold. The largest α is referred to as the convergence rate of the system.

$$\alpha_1^{\eta_1}\alpha_2^{\eta_2}\cdots\alpha_N^{\eta_N}>\alpha>1,$$

$$V\left(x(k+1)\right)-V\left(x(k)\right)\le\left(\alpha_s^{-2}-1\right)V\left(x(k)\right),\quad s=1,2,\cdots,N$$

where N is the number of events of the system. $\eta_1,\eta_2,\cdots,\eta_N$ are the rate of occurrence of these events.

Schur Complement Lemma. (Yu, 2002) Let $\boldsymbol{Z}$ be a symmetric matrix given by $\boldsymbol{Z}=\begin{bmatrix}\boldsymbol{Z}_{11} & \boldsymbol{Z}_{12}\\ \boldsymbol{Z}_{12}^{\mathrm{T}} & \boldsymbol{Z}_{22}\end{bmatrix}$, where $\boldsymbol{Z}_{11}\in\mathbf{R}^{p\times p}$. Let $\boldsymbol{Z}_S$ be the Schur Complement of $\boldsymbol{Z}_{11}$ in $\boldsymbol{Z}$, that is $\boldsymbol{Z}_S=\boldsymbol{Z}_{22}-\boldsymbol{Z}_{12}^{\mathrm{T}}\boldsymbol{Z}_{11}^{-1}\boldsymbol{Z}_{12}$, then

i) $\boldsymbol{Z}$ is negative definite if and only if $\boldsymbol{Z}_{11}$ and $\boldsymbol{Z}_S$ are both negative definite:

$\boldsymbol{Z}<0\Leftrightarrow\boldsymbol{Z}_{11}<0,\boldsymbol{Z}_S=\boldsymbol{Z}_{22}-\boldsymbol{Z}_{12}^{\mathrm{T}}\boldsymbol{Z}_{11}^{-1}\boldsymbol{Z}_{12}<0$.

ii) $\boldsymbol{Z}$ is negative definite if and only if $\boldsymbol{Z}_{22}$ and $\boldsymbol{Z}_{11}-\boldsymbol{Z}_{12}\boldsymbol{Z}_{22}^{-1}\boldsymbol{Z}_{12}^{\mathrm{T}}$ are both negative definite:

$\boldsymbol{Z}<0\Leftrightarrow\boldsymbol{Z}_{22}<0,\boldsymbol{Z}_{11}-\boldsymbol{Z}_{12}\boldsymbol{Z}_{22}^{-1}\boldsymbol{Z}_{12}^{\mathrm{T}}<0$.

Lemma 3. (Yu, 2002) Given a symmetric matrix $\boldsymbol{U}$ with 3 row partitions and 3 column partitions, there exists a matrix $\boldsymbol{V}=\boldsymbol{U}_{13}{}^{\mathrm{T}}\boldsymbol{U}_{11}{}^{-1}\boldsymbol{U}_{12}-\boldsymbol{U}_{23}{}^{\mathrm{T}}$ satisfying $\begin{bmatrix}\boldsymbol{U}_{11} & \boldsymbol{U}_{12} & \boldsymbol{U}_{13}\\ \boldsymbol{U}_{12}{}^{\mathrm{T}} & \boldsymbol{U}_{22} & \boldsymbol{U}_{23}+\boldsymbol{V}^{\mathrm{T}}\\ \boldsymbol{U}_{13}{}^{\mathrm{T}} & \boldsymbol{U}_{23}{}^{\mathrm{T}}+\boldsymbol{V} & \boldsymbol{U}_{33}\end{bmatrix}<0$ if and only if $\begin{bmatrix}\boldsymbol{U}_{11} & \boldsymbol{U}_{12}\\ \boldsymbol{U}_{12}{}^{\mathrm{T}} & \boldsymbol{U}_{22}\end{bmatrix}<0$, $\begin{bmatrix}\boldsymbol{U}_{11} & \boldsymbol{U}_{13}\\ \boldsymbol{U}_{13}{}^{\mathrm{T}} & \boldsymbol{U}_{33}\end{bmatrix}<0$.

Consensus analysis and design for MAS in case A

The next Theorem 1 reveals that the robust exponential stability of an asynchronous dynamical system subject to random short time delay, packet loss, and environmental noise can be guaranteed and the corresponding γ-suboptimal H_∞ controller gain can be obtained by the following LMIs. That is to say, the MAS in case A satisfying the conditions of Theorem 1 can reach consensus.

In the following Theorem 1 and Theorem 2, the scalars ε, α_1 and α_2 are defined in Remark 3.

Remark 3. ε is a positive scalar. α_1 and α_2 are scalars mentioned in Lemma 2 where $N=2$ indicating that there are 2 events in the asynchronous dynamical system.

Theorem 1. For an asynchronous dynamical system constrained by configuration events S_1, S_2, suppose the occurrence rate of event S_1 is η. Given a scalar $\gamma > 0$, the MAS described by (10) and (11) is robustly exponentially stable with γ noise attenuation level, the convergence rate is α, and the corresponding γ-suboptimal H_∞ controller gain matrix can be chosen as $\boldsymbol{K} = \boldsymbol{T}\boldsymbol{X}^{-1}$, if there exist scalars $\varepsilon > 0$, $1 < \alpha_2 < \alpha_1$, symmetric positive definite matrices $\boldsymbol{X} \in \mathbf{R}^{n\times n}$, $\boldsymbol{Y} \in \mathbf{R}^{n\times n}$, and matrix $\boldsymbol{T}$, such that the inequality (16) and LMIs (17), (18) hold.

$$\alpha_1^{\eta}\alpha_2^{1-\eta} > \alpha > 1 \tag{16}$$

$$\begin{bmatrix} \varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} - \boldsymbol{X} & \boldsymbol{A}_k\boldsymbol{X} + \boldsymbol{B}_0\boldsymbol{T} & \boldsymbol{B}_1\boldsymbol{T} & \boldsymbol{G}_w & \boldsymbol{0} & \boldsymbol{0} \\ \left(\boldsymbol{A}_k\boldsymbol{X} + \boldsymbol{B}_0\boldsymbol{T}\right)^{\mathrm{T}} & \boldsymbol{Y} - \alpha_1^{-2}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & \left(\boldsymbol{E}\boldsymbol{T}\right)^{\mathrm{T}} & \boldsymbol{X}\boldsymbol{C}^{\mathrm{T}} \\ \left(\boldsymbol{B}_1\boldsymbol{T}\right)^{\mathrm{T}} & \boldsymbol{0} & -\alpha_1^{-2}\boldsymbol{Y} & \boldsymbol{0} & -\left(\boldsymbol{E}\boldsymbol{T}\right)^{\mathrm{T}} & \boldsymbol{0} \\ \boldsymbol{G}_w^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} & -\gamma^2\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E}\boldsymbol{T} & -\boldsymbol{E}\boldsymbol{T} & \boldsymbol{0} & -\varepsilon\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} \end{bmatrix} < 0 \tag{17}$$

$$\begin{bmatrix} -\boldsymbol{X} & \boldsymbol{A}_k\boldsymbol{X} & (\boldsymbol{B}_0 + \boldsymbol{B}_1)\boldsymbol{T} & \boldsymbol{G}_w & \boldsymbol{0} \\ \boldsymbol{X}\boldsymbol{A}_k^{\mathrm{T}} & -\alpha_2^{-2}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{X}\boldsymbol{C}^{\mathrm{T}} \\ \left[(\boldsymbol{B}_0 + \boldsymbol{B}_1)\boldsymbol{T}\right]^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{Y} - \alpha_2^{-2}\boldsymbol{Y} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{G}_w^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} & -\gamma^2\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} \end{bmatrix} < 0 \tag{18}$$

Proof. Suppose the rate of successful data transmission is set as η ($0 < \eta \le 1$), then Lemma 2 shows that packet loss rate in the asynchronous dynamical system $\tilde{\boldsymbol{z}}_i(k+1) = \boldsymbol{\Psi}_s\tilde{\boldsymbol{z}}_i(k) + \tilde{\boldsymbol{G}}\boldsymbol{w}_i(k),\ (s = 1,2)$ is $1-\eta$. The system is exponentially stable, if inequality (16) holds. Due to $0 < \eta \le 1$, and $\alpha_1^{\eta}\alpha_2^{1-\eta} > \alpha > 1$, two cases can be obtained:

I) When $\lg\alpha_2^{-2}/(\lg\alpha_2^{-2} - \lg\alpha_1^{-2}) < 0$, $1 < \alpha_2 < \alpha_1$ or $0 < \alpha_1 < \alpha_2 < 1$ is derived;

II) When $0 < \lg\alpha_2^{-2}/(\lg\alpha_2^{-2} - \lg\alpha_1^{-2}) < 1$, $\alpha_2 > 1, 0 < \alpha_1 < 1$ or $\alpha_1 > 1, 0 < \alpha_2 < 1$ is obtained.

Thus, $1 < \alpha_2 < \alpha_1$ can be chosen to meet aftermentioned inequality (20).

Define the Lyapunov function: $V(k) = \boldsymbol{z}_i^{\mathrm{T}}(k)\boldsymbol{P}\boldsymbol{z}_i(k) + \boldsymbol{z}_i^{\mathrm{T}}(k-1)\boldsymbol{Q}\boldsymbol{z}_i(k-1)$, where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are symmetric positive definite matrices.

Under the zero-initial condition, the system H_∞ performance index function J can be described by

$$J = \sum_{k=0}^{\infty}\left[\bar{\boldsymbol{y}}_i^{\mathrm{T}}(k)\bar{\boldsymbol{y}}_i(k) - \gamma^2\boldsymbol{w}_i^{\mathrm{T}}(k)\boldsymbol{w}_i(k)\right] \tag{19}$$

If $V(k+1) - V(k) < V(k+1) - \alpha_s^{-2}V(k)$ holds, $\alpha_s > 1$ $(s = 1,2)$ must be satisfied. For all $\boldsymbol{w}_i(k) \ne 0$, $\boldsymbol{w}_i(k) \in L_2[0,\infty)$, where $L_2[0,\infty)$ is the space of square-integrable vector functions over $[0,\infty)$, $V(0) = 0$, $V(\infty) \ge 0$. The H_∞ performance of the system can be ensured, if (20) holds.

$$
\begin{aligned}
J &= \sum_{k=0}^{\infty}\left[\overline{\boldsymbol{y}}_i^{\mathrm{T}}(k)\overline{\boldsymbol{y}}_i(k)-\gamma^2\boldsymbol{w}_i^{\mathrm{T}}(k)\boldsymbol{w}_i(k)+\Delta V(k)\right]-\sum_{k=0}^{\infty}\Delta V(k) \\
&< \sum_{k=0}^{\infty}\left[\overline{\boldsymbol{y}}_i^{\mathrm{T}}(k)\overline{\boldsymbol{y}}_i(k)-\gamma^2\boldsymbol{w}_i^{\mathrm{T}}(k)\boldsymbol{w}_i(k)+\left(V(k+1)-\alpha_s^{-2}V(k)\right)\right]-\left(V(\infty)-V(0)\right) \\
&\le \sum_{k=0}^{\infty}\left[\overline{\boldsymbol{y}}_i^{\mathrm{T}}(k)\overline{\boldsymbol{y}}_i(k)-\gamma^2\boldsymbol{w}_i^{\mathrm{T}}(k)\boldsymbol{w}_i(k)+\left(V(k+1)-\alpha_s^{-2}V(k)\right)\right]<0
\end{aligned}
$$

(20)

According to the system equations (10) and (11), $V\left(k+1\right)-\alpha_s^{-2}V\left(k\right)$ ($s=1,2$) can be derived as

$$
\begin{aligned}
&V\left(k+1\right)-\alpha_s^{-2}V\left(k\right) \\
&=\begin{bmatrix}\boldsymbol{z}_i^{\mathrm{T}}(k) & \boldsymbol{z}_i^{\mathrm{T}}(k-1) & \boldsymbol{w}_i^{\mathrm{T}}(k)\end{bmatrix}\boldsymbol{\Xi}_s\begin{bmatrix}\boldsymbol{z}_i(k)\\ \boldsymbol{z}_i(k-1)\\ \boldsymbol{w}_i(k)\end{bmatrix}
\end{aligned}
\tag{21}
$$

where $\boldsymbol{\Xi}_1=\begin{bmatrix}\boldsymbol{\Omega}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\Omega}+\boldsymbol{Q}-\alpha_1^{-2}\boldsymbol{P} & \boldsymbol{\Omega}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{S} & \boldsymbol{\Omega}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{G}_w\\ \boldsymbol{S}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\Omega} & \boldsymbol{S}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{S}-\alpha_1^{-2}\boldsymbol{Q} & \boldsymbol{S}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{G}_w\\ \boldsymbol{G}_w^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\Omega} & \boldsymbol{G}_w^{\mathrm{T}}\boldsymbol{P}\boldsymbol{S} & \boldsymbol{G}_w^{\mathrm{T}}\boldsymbol{P}\boldsymbol{G}_w\end{bmatrix}$, $\boldsymbol{\Omega}=\boldsymbol{A}_k+\left(\boldsymbol{B}_0+\boldsymbol{D}\boldsymbol{F}(\tau)\boldsymbol{E}\right)\boldsymbol{K}$, $\boldsymbol{S}=\left(\boldsymbol{B}_1-\boldsymbol{D}\boldsymbol{F}(\tau)\boldsymbol{E}\right)\boldsymbol{K}$.

$$
\boldsymbol{\Xi}_2=\begin{bmatrix}\boldsymbol{A}_k^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}_k-\alpha_2^{-2}\boldsymbol{P} & \boldsymbol{A}_k^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{B}_0+\boldsymbol{B}_1)\boldsymbol{K} & \boldsymbol{A}_k^{\mathrm{T}}\boldsymbol{P}\boldsymbol{G}_w\\ \left[(\boldsymbol{B}_0+\boldsymbol{B}_1)\boldsymbol{K}\right]^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}_k & \left[(\boldsymbol{B}_0+\boldsymbol{B}_1)\boldsymbol{K}\right]^{\mathrm{T}}\boldsymbol{P}\left[(\boldsymbol{B}_0+\boldsymbol{B}_1)\boldsymbol{K}\right]+\boldsymbol{Q}-\alpha_2^{-2}\boldsymbol{Q} & \left[(\boldsymbol{B}_0+\boldsymbol{B}_1)\boldsymbol{K}\right]^{\mathrm{T}}\boldsymbol{P}\boldsymbol{G}_w\\ \boldsymbol{G}_w^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}_k & \boldsymbol{G}_w^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{B}_0+\boldsymbol{B}_1)\boldsymbol{K} & \boldsymbol{G}_w^{\mathrm{T}}\boldsymbol{P}\boldsymbol{G}_w\end{bmatrix}.
$$

Substituting $\overline{\boldsymbol{y}}_i(k)=\boldsymbol{C}\boldsymbol{z}_i(k)$ and (21) into (20), then

$$
\begin{aligned}
&\overline{\boldsymbol{y}}_i^{\mathrm{T}}(k)\overline{\boldsymbol{y}}_i(k)-\gamma^2\boldsymbol{w}_i^{\mathrm{T}}(k)\boldsymbol{w}_i(k)+\left(V(k+1)-\alpha_s^{-2}V(k)\right) \\
&=\begin{bmatrix}\boldsymbol{z}_i^{\mathrm{T}}(k) & \boldsymbol{z}_i^{\mathrm{T}}(k-1) & \boldsymbol{w}_i^{\mathrm{T}}(k)\end{bmatrix}\boldsymbol{\Xi}_{ss}\begin{bmatrix}\boldsymbol{z}_i(k)\\ \boldsymbol{z}_i(k-1)\\ \boldsymbol{w}_i(k)\end{bmatrix}
\end{aligned}
\tag{22}
$$

where $\boldsymbol{\Xi}_{ss}=\boldsymbol{\Xi}_s+\begin{bmatrix}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & -\gamma^2\boldsymbol{I}\end{bmatrix}$, $s=1,2$.

Using Schur Complement Lemma and Lemma 1, it can be easily verified that (23) and (24) ensure that $\boldsymbol{\Xi}_{ss}<0$, i.e. $J<0$.

$$
\begin{bmatrix}\varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}}-\boldsymbol{P}^{-1} & \boldsymbol{A}_k+\boldsymbol{B}_0\boldsymbol{K} & \boldsymbol{B}_1\boldsymbol{K} & \boldsymbol{G}_w & \boldsymbol{0}\\ \left(\boldsymbol{A}_k+\boldsymbol{B}_0\boldsymbol{K}\right)^{\mathrm{T}} & \boldsymbol{Q}-\alpha_1^{-2}\boldsymbol{P}+\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} & \left(\boldsymbol{E}\boldsymbol{K}\right)^{\mathrm{T}}\\ \left(\boldsymbol{B}_1\boldsymbol{K}\right)^{\mathrm{T}} & \boldsymbol{0} & -\alpha_1^{-2}\boldsymbol{Q} & \boldsymbol{0} & -\left(\boldsymbol{E}\boldsymbol{K}\right)^{\mathrm{T}}\\ \boldsymbol{G}_w^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} & -\gamma^2\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{E}\boldsymbol{K} & -\boldsymbol{E}\boldsymbol{K} & \boldsymbol{0} & -\varepsilon\boldsymbol{I}\end{bmatrix}<0
\tag{23}
$$

$$
\begin{bmatrix}
-\boldsymbol{P}^{-1} & \boldsymbol{A}_k & (\boldsymbol{B}_0+\boldsymbol{B}_1)\boldsymbol{K} & \boldsymbol{G}_w \\
\boldsymbol{A}_k^{\mathrm{T}} & -\alpha_2^{-2}\boldsymbol{P}+\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} \\
\left[(\boldsymbol{B}_0+\boldsymbol{B}_1)\boldsymbol{K}\right]^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{Q}-\alpha_2^{-2}\boldsymbol{Q} & \boldsymbol{0} \\
\boldsymbol{G}_w^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} & -\gamma^2\boldsymbol{I}
\end{bmatrix} < 0 \tag{24}
$$

From Lemma 3, it can be verified that when $\boldsymbol{w}(k) \equiv 0$ in (10) and (11), the exponential stability of the system can also be ensured from (23) and (24).

Premultiplying and postmultiplying (23) by $diag\left\{\boldsymbol{I}, \boldsymbol{P}^{-1}, \boldsymbol{P}^{-1}, \boldsymbol{I}, \boldsymbol{I}\right\}$ respectively, and setting $\boldsymbol{X} = \boldsymbol{P}^{-1}, \boldsymbol{Y}=\boldsymbol{P}^{-1}\boldsymbol{Q}\boldsymbol{P}^{-1}, \boldsymbol{T} = \boldsymbol{K}\boldsymbol{P}^{-1}$. According to Schur Complement Lemma, (23) can be converted to (17).

Premultiplying and postmultiplying (24) by $diag\left\{\boldsymbol{I}, \boldsymbol{P}^{-1}, \boldsymbol{P}^{-1}, \boldsymbol{I}\right\}$ respectively, and make $\boldsymbol{X} = \boldsymbol{P}^{-1}, \boldsymbol{Y}=\boldsymbol{P}^{-1}\boldsymbol{Q}\boldsymbol{P}^{-1}, \boldsymbol{T} = \boldsymbol{K}\boldsymbol{P}^{-1}$. According to Schur Complement Lemma, (24) can be converted to (18).

This completes the proof of Theorem 1. The MAS described by (10) and (11) is exponentially stable with γ noise attenuation level. And the γ-suboptimal H_∞ controller gain matrix guaranteeing robust exponential stability of system can be chosen as $\boldsymbol{K} = \boldsymbol{T}\boldsymbol{X}^{-1}$, because $\boldsymbol{X} = \boldsymbol{P}^{-1}$ and $\boldsymbol{T} = \boldsymbol{K}\boldsymbol{P}^{-1}$.

In conclusion, the MAS in case A satisfying the conditions of Theorem 1 can reach consensus.

Consensus analysis and design for MAS in case B

The following Theorem 2 shows that the robust exponential stability of the MAS modeled as an asynchronous dynamical system subject to random long time delay, packet loss, and environmental noise can be guaranteed and the corresponding H_∞ control is also achieved in terms of the following matrix inequalities. Theorem 2 provides the conditions for the solvability of the MAS consensus in case B.

Theorem 2. For an asynchronous dynamical system constrained by configuration events S_1, S_2, suppose the occurrence rate of event S_1 is η. Given a scalar $\gamma > 0$, the MAS described by (14) and (15) is robustly exponentially stable with γ noise attenuation level, the convergence rate is α, and the corresponding γ-suboptimal H_∞ controller gain matrix can be chosen as $\boldsymbol{K} = \boldsymbol{T}\boldsymbol{X}^{-1}$, if there exist scalars $\varepsilon > 0$, $1 < \alpha_2 < \alpha_1$, symmetric positive definite matrix $\boldsymbol{X} \in \mathbf{R}^{n\times n}$, and matrix $\boldsymbol{T}$, such that inequality (25) and LMIs (26), (27) hold.

$$
\alpha_1^{\eta}\alpha_2^{1-\eta} > \alpha > 1 \tag{25}
$$

$$
\begin{bmatrix}
\varepsilon\hat{\boldsymbol{D}}\hat{\boldsymbol{D}}^{\mathrm{T}}-\boldsymbol{X} & \boldsymbol{H}\boldsymbol{X}+\boldsymbol{\Theta}\boldsymbol{T} & \hat{\boldsymbol{G}} & \boldsymbol{0} & \boldsymbol{0} \\
(\boldsymbol{H}\boldsymbol{X}+\boldsymbol{\Theta}\boldsymbol{T})^{\mathrm{T}} & -\alpha_1^{-2}\boldsymbol{X} & \boldsymbol{0} & (\hat{\boldsymbol{E}}\boldsymbol{X})^{\mathrm{T}} & \boldsymbol{X}\hat{\boldsymbol{C}}^{\mathrm{T}} \\
\hat{\boldsymbol{G}}^{\mathrm{T}} & \boldsymbol{0} & -\gamma^2\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & \hat{\boldsymbol{E}}\boldsymbol{X} & \boldsymbol{0} & -\varepsilon\boldsymbol{I} & \boldsymbol{0} \\
\boldsymbol{0} & \hat{\boldsymbol{C}}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}
\end{bmatrix} < 0 \tag{26}
$$

$$\begin{bmatrix} -\boldsymbol{X} & \boldsymbol{\Psi}_2\boldsymbol{X} & \hat{\boldsymbol{G}} & \boldsymbol{0} \\ \boldsymbol{X}\boldsymbol{\Psi}_2^{\mathrm{T}} & -\alpha_2^{-2}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{X}\hat{\boldsymbol{C}}^{\mathrm{T}} \\ \hat{\boldsymbol{G}}^{\mathrm{T}} & \boldsymbol{0} & -\gamma^2\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \hat{\boldsymbol{C}}\boldsymbol{X} & \boldsymbol{0} & -\boldsymbol{I} \end{bmatrix} < 0 \tag{27}$$

Proof. Similar to Theorem 1, inequality (25) holds.

Define the Lyapunov function: $V(k) = \hat{\boldsymbol{z}}_i^{\mathrm{T}}(k)\boldsymbol{P}\hat{\boldsymbol{z}}_i(k)$, where $\boldsymbol{P}$ is symmetric positive definite matrix.

Under the zero-initial condition, the system H_∞ performance index J can also be described as (19), where $\bar{\boldsymbol{y}}_i(k)$ is replaced by $\hat{\boldsymbol{y}}_i(k)$.

According to the system equations (14) and (15), $V(k+1) - \alpha_s^{-2}V(k)$ ($s = 1,2$) can be derived as

$$\begin{aligned} & V(k+1) - \alpha_s^{-2}V(k) \\ & = \begin{bmatrix} \hat{\boldsymbol{z}}_i^{\mathrm{T}}(k) & \boldsymbol{w}_i^{\mathrm{T}}(k) \end{bmatrix} \begin{bmatrix} \boldsymbol{\Psi}_s^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\Psi}_s - \alpha_s^{-2}\boldsymbol{P} & \boldsymbol{\Psi}_s^{\mathrm{T}}\boldsymbol{P}\hat{\boldsymbol{G}} \\ \hat{\boldsymbol{G}}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\Psi}_s & \hat{\boldsymbol{G}}^{\mathrm{T}}\boldsymbol{P}\hat{\boldsymbol{G}} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{z}}_i(k) \\ \boldsymbol{w}_i(k) \end{bmatrix} \end{aligned} \tag{28}$$

Similar to the proof of Theorem 1, substituting $\hat{\boldsymbol{y}}_i(k)=\hat{\boldsymbol{C}}\hat{\boldsymbol{z}}_i(k)$ and (28) into $J < 0$, then according to Schur Complement Lemma, (29) can be obtained

$$\begin{bmatrix} -\boldsymbol{P}^{-1} & \boldsymbol{\Psi}_s & \hat{\boldsymbol{G}} \\ \boldsymbol{\Psi}_s^{\mathrm{T}} & -\alpha_s^{-2}\boldsymbol{P} + \hat{\boldsymbol{C}}^{\mathrm{T}}\hat{\boldsymbol{C}} & \boldsymbol{0} \\ \hat{\boldsymbol{G}}^{\mathrm{T}} & \boldsymbol{0} & -\gamma^2\boldsymbol{I} \end{bmatrix} < 0 \tag{29}$$

Substituting $\boldsymbol{\Psi}_1 = \boldsymbol{H} + \hat{\boldsymbol{D}}\boldsymbol{F}(\tau_k{}')\hat{\boldsymbol{E}} + \boldsymbol{\Theta}\boldsymbol{K}$ into (29), and according to Schur Complement Lemma and Lemma 1, (30) can be obtained

$$\begin{bmatrix} \varepsilon\hat{\boldsymbol{D}}\hat{\boldsymbol{D}}^{\mathrm{T}} - \boldsymbol{P}^{-1} & \boldsymbol{H} + \boldsymbol{\Theta}\boldsymbol{K} & \hat{\boldsymbol{G}} & \boldsymbol{0} \\ (\boldsymbol{H} + \boldsymbol{\Theta}\boldsymbol{K})^{\mathrm{T}} & -\alpha_1^{-2}\boldsymbol{P} + \hat{\boldsymbol{C}}^{\mathrm{T}}\hat{\boldsymbol{C}} & \boldsymbol{0} & \hat{\boldsymbol{E}}^{\mathrm{T}} \\ \hat{\boldsymbol{G}}^{\mathrm{T}} & \boldsymbol{0} & -\gamma^2\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \hat{\boldsymbol{E}} & \boldsymbol{0} & -\varepsilon\boldsymbol{I} \end{bmatrix} < 0 \tag{30}$$

Substituting $\boldsymbol{\Psi}_2$ into (29), then (31) is as follows

$$\begin{bmatrix} -\boldsymbol{P}^{-1} & \boldsymbol{\Psi}_2 & \hat{\boldsymbol{G}} \\ \boldsymbol{\Psi}_2^{\mathrm{T}} & -\alpha_2^{-2}\boldsymbol{P} + \hat{\boldsymbol{C}}^{\mathrm{T}}\hat{\boldsymbol{C}} & \boldsymbol{0} \\ \hat{\boldsymbol{G}}^{\mathrm{T}} & \boldsymbol{0} & -\gamma^2\boldsymbol{I} \end{bmatrix} < 0 \tag{31}$$

From Lemma 3, it can be verified that when $\boldsymbol{w}(k) \equiv 0$ in (14) and (15), the exponential stability of the system can also be ensured from (30) and (31).

Premultiplying and postmultiplying (30) by $diag\left\{\boldsymbol{I}, \boldsymbol{P}^{-1}, \boldsymbol{I}, \boldsymbol{I}\right\}$, and setting $\boldsymbol{X} = \boldsymbol{P}^{-1}, \boldsymbol{T} = \boldsymbol{K}\boldsymbol{P}^{-1}$, according to Schur Complement Lemma, well then, (26) in Theorem 2 holds.

Premultiplying and postmultiplying (31) by $diag\left\{\boldsymbol{I}, \boldsymbol{P}^{-1}, \boldsymbol{I}\right\}$, and setting $\boldsymbol{X} = \boldsymbol{P}^{-1}$, well then, (27) in Theorem 2 holds.

This completes the proof of Theorem 2. The MAS described by (14) and (15) is robustly exponentially stable with γ noise attenuation level, and the corresponding γ-suboptimal H_∞ controller gain can be designed as $\boldsymbol{K} = \boldsymbol{T}\boldsymbol{X}^{-1}$, because $\boldsymbol{X} = \boldsymbol{P}^{-1}$ and $\boldsymbol{T} = \boldsymbol{K}\boldsymbol{P}^{-1}$, if the conditions of Theorem 2 are satisfied.

In conclusion, the MAS in case B satisfying the conditions of Theorem 2 can reach consensus.

Remark 4. The convergence rate of the system can be determined, if a set of scalars $\alpha_s > 0\,(s = 1, 2)$ satisfying $\alpha_1^{\eta}\alpha_2^{1-\eta} > \alpha > 1$ are selected and the packet loss rate is given. The calculated convergence rate of the system will differ, if the values of α_1 and α_2 are chosen differently. The LMIs (17) and (18) or (26) and (27) can be solved by using the Matlab LMI toolbox, and two conclusions are obtained depending on whether there are feasible solutions to the matrix inequalities: the system will not reach robustly exponentially stable state, if there are no feasible solutions; the system will reach robustly exponentially stable state and the corresponding controller gain can be further determined, if there are feasible solutions.

Admissible packet loss probability

The packet loss probability of network is vital to stability of MAS modeled as asynchronous dynamical system. By the previous Theorems, the inequality $\alpha_1^{\eta}\alpha_2^{1-\eta} > \alpha > 1$ is satisfied. Combining with Lemma 2, the admissible packet loss probability guaranteeing the robust exponential stability of the system can be calculated by the following Corollary.

Corollary. For the MAS described in Theorems 1 and 2, the admissible packet loss probability guaranteeing the robust exponential stability of the system is as follows, if conditions of Theorems are satisfied:

i) The closed-loop system with communication constraints is robustly exponentially stable for $0 < \eta \le 1$, if the open-loop system is stable (including critical stability).

ii) The closed-loop system with communication constraints is robustly exponentially stable for $1/(1-\mu_1/\mu_2) < \eta \le 1$, $\left(\mu_1 = \lg\left[\lambda^2_{\max}(\boldsymbol{\Psi}_1)\right], \mu_2 = \lg\left[\lambda^2_{\max}(\boldsymbol{\Psi}_2)\right]\right)$, if the open-loop system is not stable.

Before proceeding, the following Lemma that will be used for the proof of Corollary is considered.

Lemma 4. (Fridman and Shaked, 2005) $\boldsymbol{A} \in \mathbf{R}^{n\times n}$ is a real symmetric positive definite matrix and $\boldsymbol{S} \in \mathbf{R}^{n\times n}$ is a real matrix, the spectral radius of $\boldsymbol{S}$, $\rho(\boldsymbol{S}) < 1$, if $\boldsymbol{A} - \boldsymbol{S}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{S} > 0$.

Proof. i) Let $\delta_s = \alpha_s^{-2}$, ($s = 1, 2$). It can be deduced from Lemma 2 that the system $\tilde{\boldsymbol{z}}_i(k+1) = \boldsymbol{\Psi}_s\tilde{\boldsymbol{z}}_i(k) + \tilde{\boldsymbol{G}}\boldsymbol{w}(k)$ is robustly exponentially stable, if $\boldsymbol{\Psi}_s^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\Psi}_s - \alpha_s^{-2}\boldsymbol{P} < 0$, i.e. $\left(\boldsymbol{\Psi}_s\dfrac{1}{\sqrt{\delta_s}}\right)^{\mathrm{T}}\boldsymbol{P}\left(\boldsymbol{\Psi}_s\dfrac{1}{\sqrt{\delta_s}}\right) - \boldsymbol{P} < 0$. Therefore, the value of δ_s ensuring the stability of the system should satisfy $\left|\lambda_{\max}\left(\boldsymbol{\Psi}_s\dfrac{1}{\sqrt{\delta_s}}\right)\right| < 1$, i.e., $\delta_s > \lambda^2_{\max}(\boldsymbol{\Psi}_s)$.

$\frac{\lg\delta_2}{\lg\delta_2 - \lg\delta_1} < \eta \le 1$ is obvious, because $\alpha_1^{\eta}\alpha_2^{1-\eta} > \alpha > 1$, $0 < \eta \le 1$.

If the open-loop system is stable (including critical stability), i.e. $|\lambda(\boldsymbol{A}_k)| \le 1$, the closed-loop system is stable ($|\lambda_{\max}(\boldsymbol{\Psi}_s)| = 1$), and with the condition $\delta_s \ge \lambda_{\max}^2(\boldsymbol{\Psi}_s)$ and $\frac{\lg\delta_2}{\lg\delta_2 - \lg\delta_1} < \eta \le 1$, then the closed-loop system with communication constraints is robustly exponentially stable for $0 < \eta \le 1$.

ii) If the open-loop system is not stable, let $\mu_s = \min(\lg\delta_s) = \lg\left[\lambda_{\max}^2(\boldsymbol{\Psi}_s)\right]$, ($s = 1, 2$) and $\frac{\lg\delta_2}{\lg\delta_2 - \lg\delta_1} < \eta \le 1$, so the closed-loop system with communication constraints is robustly exponentially stable for $1/(1-\mu_1/\mu_2) < \eta \le 1$.

The proof of Corollary is completed.

In conclusion, the MAS can reach consensus, if the packet loss probability satisfies the conditions of corollary.

Remark 5. For the MAS described by (7), under the effect of the γ-suboptimal H_∞ controllers designed in Theorems 1 and 2, the impacts of different time delays and packet loss probabilities on the consensus of MAS with environmental noise will be analyzed in the next section.

Simulation & Verification of MAS consensus

The double wheeled differential mobile robot is taken as an agent example (Liu et al., 2009) to verify the proposed methods. The physical model of this kind of robot is shown in Figure 3.

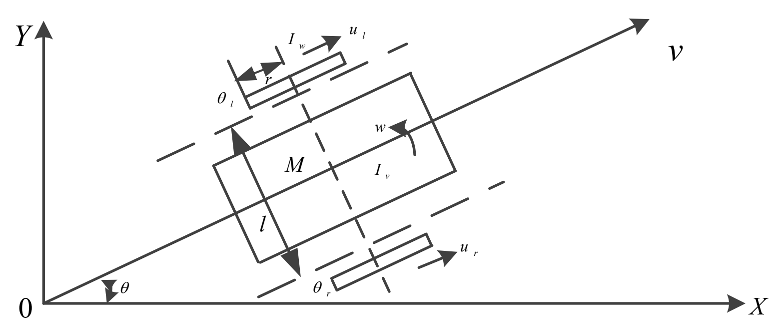


Figure 3. The physical model diagram of the double wheeled differential mobile robot

As shown in Figure 3, I_v and I_w are denoted as the moment of inertia around the center of gravity of the robot and the moment of inertia of wheel, respectively. M is the mass of robot. r is the wheel radius and l is the distance between the left or right wheel and the center of gravity of robot. v, θ, and w are the linear velocity, azimuth, and angular velocity of the robot, respectively. k is the driving gain factor, c is set as the viscous friction factor between the robot and the ground. θ_l and θ_r are rotational angle of the left and right wheels. u_l and u_r are the driving (control) input of the left and right wheels.

From the consideration on the kinematics of the robot, the following relationships are existed between various variables of the robot:

$$
\begin{aligned}
\dot{v} &= -\frac{2c}{Mr^2+2I_w}\cdot v+\frac{kr}{Mr^2+2I_w}\cdot\left(u_r+u_l\right) \\
\dot{\theta} &= w \\
\dot{w} &= -\frac{2cl^2}{I_v r^2+2I_w l^2}\cdot\dot{\theta}+\frac{krl}{I_v r^2+2I_w l^2}\cdot\left(u_r-u_l\right)
\end{aligned} \tag{32}
$$

Let $\boldsymbol{x}_i=[v\quad\theta\quad w]^{\mathrm{T}}$ is the state vector of robot i, $\boldsymbol{u}_i=[u_r\quad u_l]^{\mathrm{T}}$ is the control vector of robot i, and $\boldsymbol{y}_i=[v\quad\theta\quad w]^{\mathrm{T}}$ is the output vector of robot i, $i=1,2,\cdots,n$. Then the continuous-time dynamics of robot i with the environmental noise can be concluded as:

$$
\begin{cases}
\dot{\boldsymbol{x}}_i(t)=\boldsymbol{A}\boldsymbol{x}_i(t)+\boldsymbol{B}\boldsymbol{u}_i(t)+\boldsymbol{G}\boldsymbol{\xi}_i(t) \\
\boldsymbol{y}_i(t)=\boldsymbol{C}\boldsymbol{x}_i(t)
\end{cases} \tag{33}
$$

where $\boldsymbol{A}=\begin{bmatrix} -\frac{2c}{Mr^2+I_w} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{2cl^2}{I_v r^2+2I_w l^2} \end{bmatrix}$, $\boldsymbol{B}=\begin{bmatrix} \frac{kr}{Mr^2+2I_w} & \frac{kr}{Mr^2+2I_w} \\ 0 & 0 \\ \frac{krl^2}{I_v r^2+2I_w l^2} & -\frac{krl^2}{I_v r^2+2I_w l^2} \end{bmatrix}$, $\boldsymbol{C}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\boldsymbol{G}=\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$.

In practical simulation conditions, the physical parameters (Liu et al., 2009) of the robot are chosen as: $I_v=10\mathrm{kg}\cdot\mathrm{m}^2$, $M=200\mathrm{kg}$, $l=0.3\mathrm{m}$, $I_w=0.005\mathrm{kg}\cdot\mathrm{m}^2$, $r=0.1\mathrm{m}$, $k=5$, $c=0.05\mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}$. Substituting these parameters into (33), (34) can be obtained

$$
\begin{cases}
\dot{\boldsymbol{x}}_i(t)=\begin{bmatrix} -1/2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}\boldsymbol{x}_i(t)+\begin{bmatrix} 1/4 & 1/4 \\ 0 & 0 \\ 1/2 & -1/2 \end{bmatrix}\boldsymbol{u}_i(t)+\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}\boldsymbol{\xi}_i(t) \\
\boldsymbol{y}_i(t)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\boldsymbol{x}_i(t)
\end{cases} \tag{34}
$$

Without loss of generality, $n=5$ is set. Suppose the fixed communication topology of the MAS is a complete graph, as shown in Figure 4. The adjacency matrix and Laplacian matrix of the MAS can be obtained from the definition in the second section, and $\tilde{\boldsymbol{L}}$ also can be calculated from the definition in (5).

$\boldsymbol{A}_l=\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$, $\boldsymbol{L}=\begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}$, $\tilde{\boldsymbol{L}}=\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$, the eigenvalues of $\tilde{\boldsymbol{L}}$ are $\lambda_2=\lambda_3=\lambda_4=\lambda_5=5$.

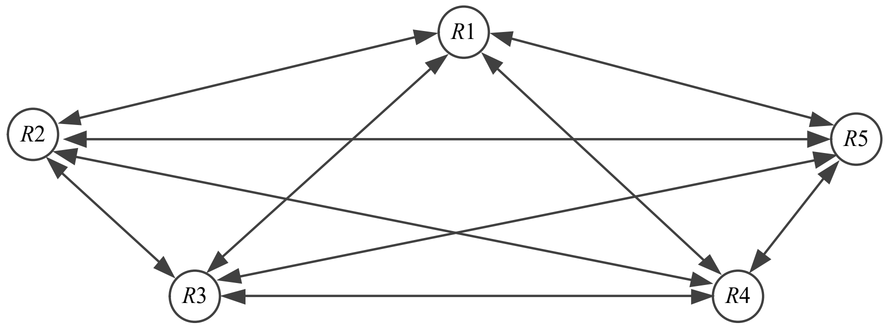


Figure 4. The communication topology graph including 5 robots

In the light of the model transformation in the third section, from (7), the system can be expressed as

$$\begin{cases} \dot{\boldsymbol{z}}_i(t) = \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \boldsymbol{z}_i(t) + \begin{bmatrix} -5/4 & -5/4 \\ 0 & 0 \\ -5/2 & 5/2 \end{bmatrix} \boldsymbol{K}\boldsymbol{z}_i(t-\tau_k) + \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \boldsymbol{w}_i(t) \\ \boldsymbol{u}_i(t) = \boldsymbol{K}\boldsymbol{z}_i(t-\tau_k) \\ \bar{\boldsymbol{y}}_i(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{z}_i(t) \end{cases}, \quad i = 2,3,4 \tag{35}$$

Let the sampling period be $T = 0.01\text{s}$, so the random short time delay is $0 < \tau_k \le 0.01\text{s}$. According to (8), (35) can be discretized as: $\boldsymbol{A}_k = \begin{bmatrix} 0.995 & 0 & 0 \\ 0 & 1 & 0.01 \\ 0 & 0 & 0.99 \end{bmatrix}$, $\boldsymbol{B}_k = \begin{bmatrix} -0.0125 & -0.0125 \\ -0.0001 & 0.0001 \\ -0.0249 & 0.0249 \end{bmatrix}$.

Let $a_1 = 1$, $a_2 = a_3 = 0$, since constants a_1, a_2 and a_3 should ensure $e^{\lambda_{mi}(T-\tau_k-a_i)} < 1, i = 1,2,3$ to satisfy $\boldsymbol{F}^{\mathrm{T}}(\tau_k)\boldsymbol{F}(\tau_k) \le \boldsymbol{I}$, then

$$\boldsymbol{B}_0 = \begin{bmatrix} -0.0125 & -0.0125 \\ -2.5000 & 2.5000 \\ -2.5000 & 2.5000 \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} 0 & 0 \\ 2.4999 & -2.4999 \\ 2.4751 & -2.4751 \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0.7071 \\ 0 & 0 & -0.7071 \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} -1.2500 & -1.2500 \\ -2.5000 & 2.5000 \\ -3.5355 & 3.5355 \end{bmatrix},$$

$$\boldsymbol{F}(\tau_k) = \begin{bmatrix} -\tau_k & 0 & 0 \\ 0 & e^{-0.5(0.01-\tau_k)} & 0 \\ 0 & 0 & e^{-(0.01-\tau_k)} \end{bmatrix}, \quad \boldsymbol{G}_w = \begin{bmatrix} 0.01 & 0.02 \\ 0 & 0 \\ 0.01 & 0.01 \end{bmatrix}.$$

It is clear that the open loop MAS is critically stable, because $\left|\lambda\left(\boldsymbol{A}_k\right)\right| \le 1$, ($\lambda_1\left(\boldsymbol{A}_k\right) = 0.995$, $\lambda_2\left(\boldsymbol{A}_k\right) = 1$, $\lambda_3\left(\boldsymbol{A}_k\right) = 0.99$). It can be obtained from the Corollary in the fifth section that the MAS (35) is robustly exponentially stable for $0 < \eta \le 1$.

In general communication system, the environmental noise spectrum is evenly distributed. Therefore, without loss of generality, it is considered as white noise which is much closer to the real communication environment (Murdock and Rappaport, 2014), as shown in Figure 5.

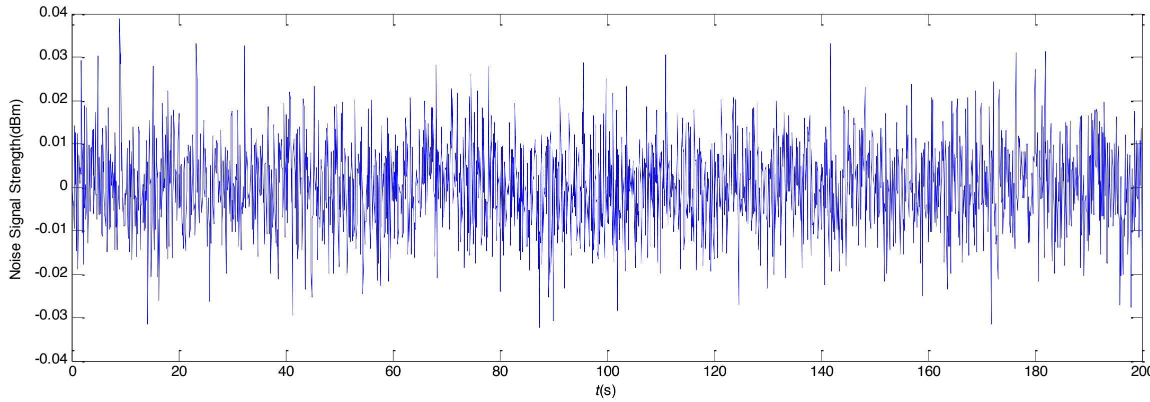


Figure 5. Environmental noise signal in the wireless network

The initial states of the five robots are given randomly. Without loss of generality, the initial states of five robots are given as $\boldsymbol{x}_1(0)=\begin{bmatrix}1 & 0 & 0\end{bmatrix}^{\mathrm{T}}$, $\boldsymbol{x}_2(0)=\begin{bmatrix}3 & -2 & 4\end{bmatrix}^{\mathrm{T}}$, $\boldsymbol{x}_3(0)=\begin{bmatrix}2 & 3 & -1\end{bmatrix}^{\mathrm{T}}$, $\boldsymbol{x}_4(0)=\begin{bmatrix}0 & 1 & 2\end{bmatrix}^{\mathrm{T}}$, $\boldsymbol{x}_5(0)=\begin{bmatrix}5 & 2 & 3\end{bmatrix}^{\mathrm{T}}$.

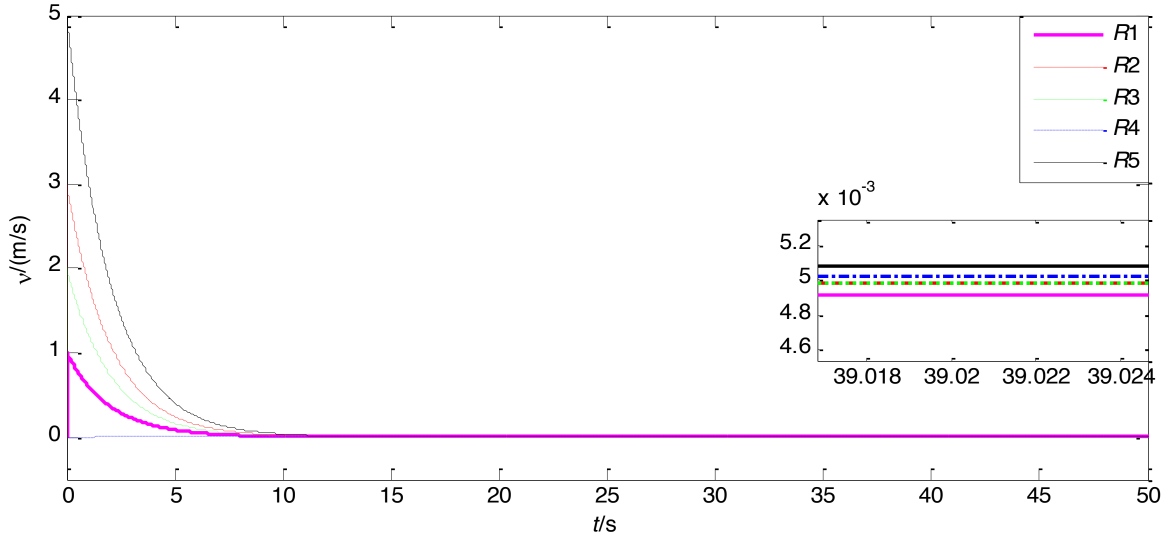
The consensus algorithm verification in case A

In order to verify the consensus algorithm in case A, i.e. verify the effectiveness of Theorem 1, the simulation is conducted for the running of five robots with the fixed communication topology shown as Figure 4, in the case A that communication constraints including random short delay, packet loss and environmental noise are considered.

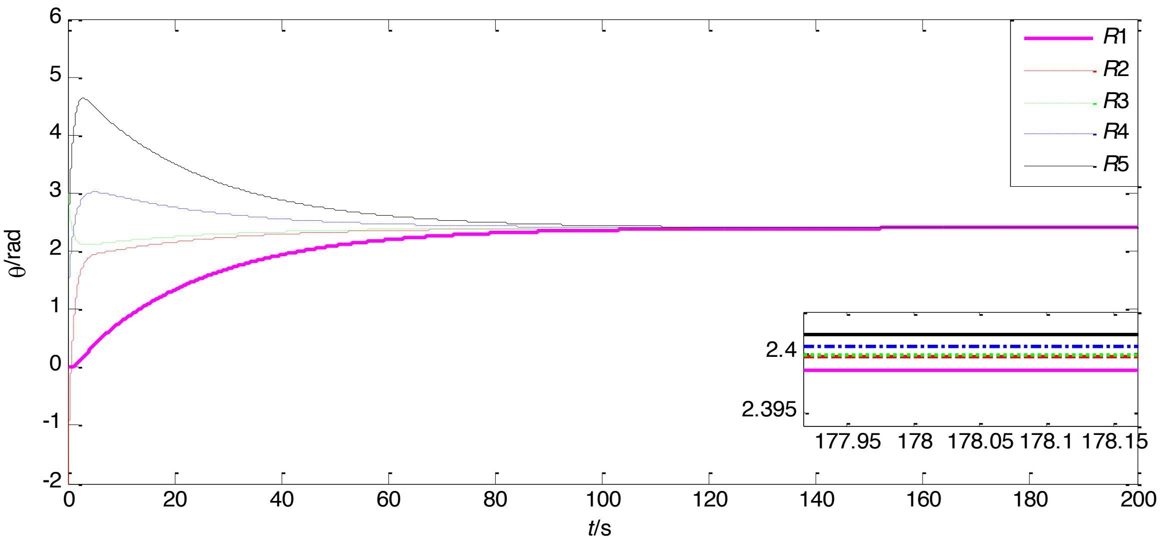
Let $\gamma=1$, and select the values of α_1 and α_2 satisfying (16) as $\alpha_1 = 1.3$, $\alpha_2 = 1.1$, the admissible packet loss probability guaranteeing the robust exponential stability of the system (35) is $0 \le 1-\eta < 1$. Suppose the successful transmission rate of data is set as $\eta = 90\%$ ($0 < \eta \le 1$), then the LMIs (17) and (18) can be solved using the Matlab LMI toolbox. The convergence rate is $\alpha = 1.2785 > 1$. The controller gain matrix can be obtained from Theorem 1 and is shown below:

$$\boldsymbol{K} = \begin{bmatrix} 0.0110 & 0.0085 & 0.0175 \\ -0.0075 & -0.0085 & -0.0160 \end{bmatrix}.$$

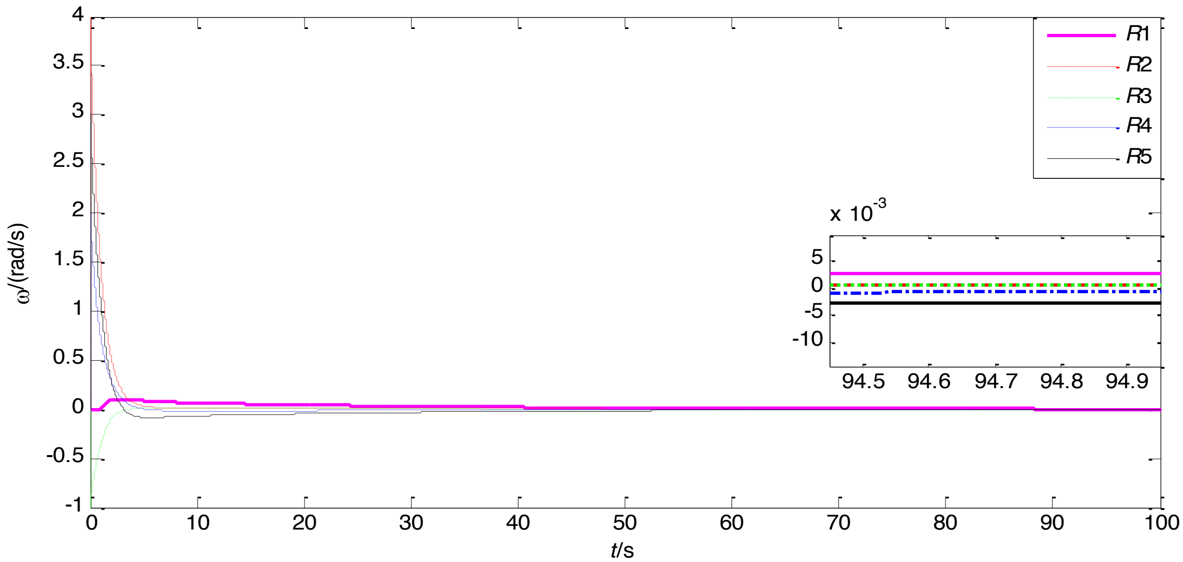
When random short delay ($0 < \tau_k \le 0.01\mathrm{s}$), packet loss rate ($1-\eta = 10\%$) and environmental noise shown in Figure 5 exist in the MAS described in (35), under the effect of the γ-suboptimal H_∞ controller obtained by solving LMIs in Theorem 1, the corresponding simulation results are shown in Figure 6. When packet loss rate is ($1-\eta = 90\%$) and other simulation conditions remain unchanged, the corresponding simulation results are given in Figure 7.



a



b



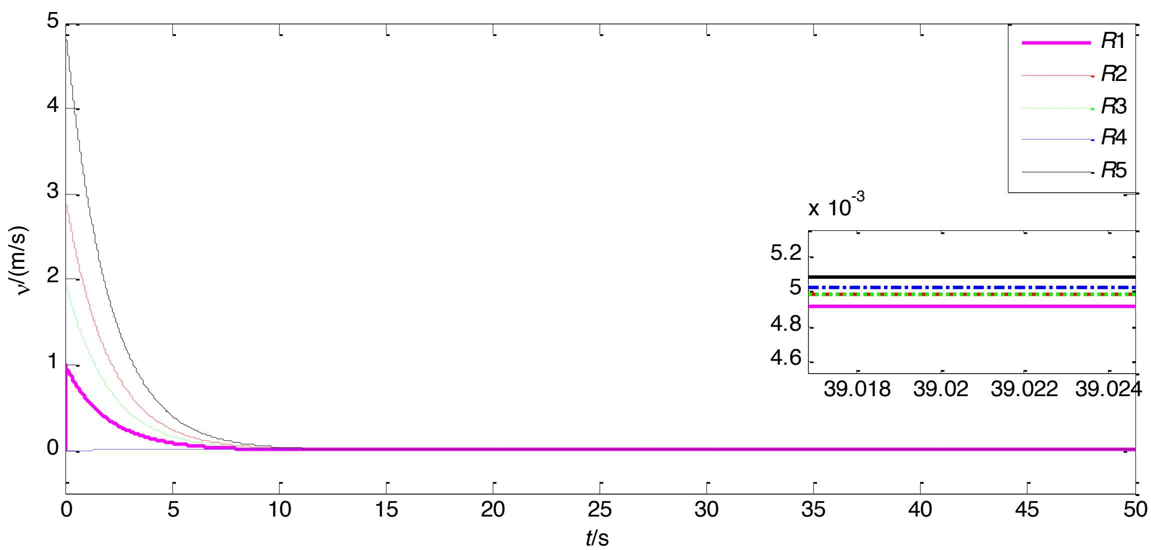
c

Figure 6. state changes of the 5 robots ($0 < \tau_k \le 0.01\mathrm{s}$, $\eta = 90\%$, white noise)

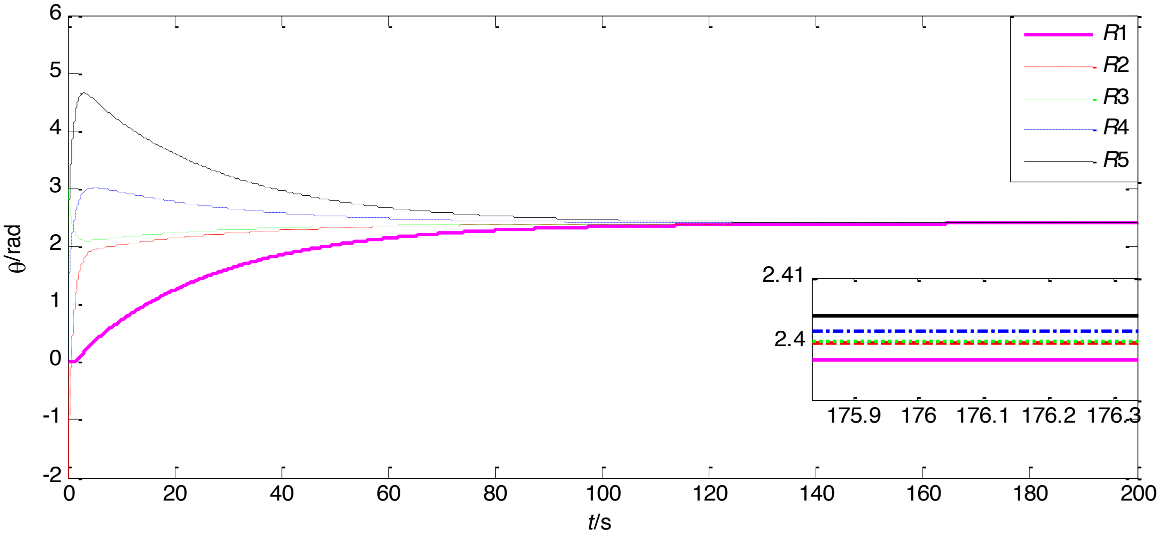
a line velocity changes of the 5 robots ($0 < \tau_k \le 0.01\mathrm{s}$, $\eta = 90\%$, white noise)

b azimuth changes of the 5 robots ($0 < \tau_k \le 0.01\mathrm{s}$, $\eta = 90\%$, white noise)

c angular velocity changes of the 5 robots ($0 < \tau_k \le 0.01\mathrm{s}$, $\eta = 90\%$, white noise)



a



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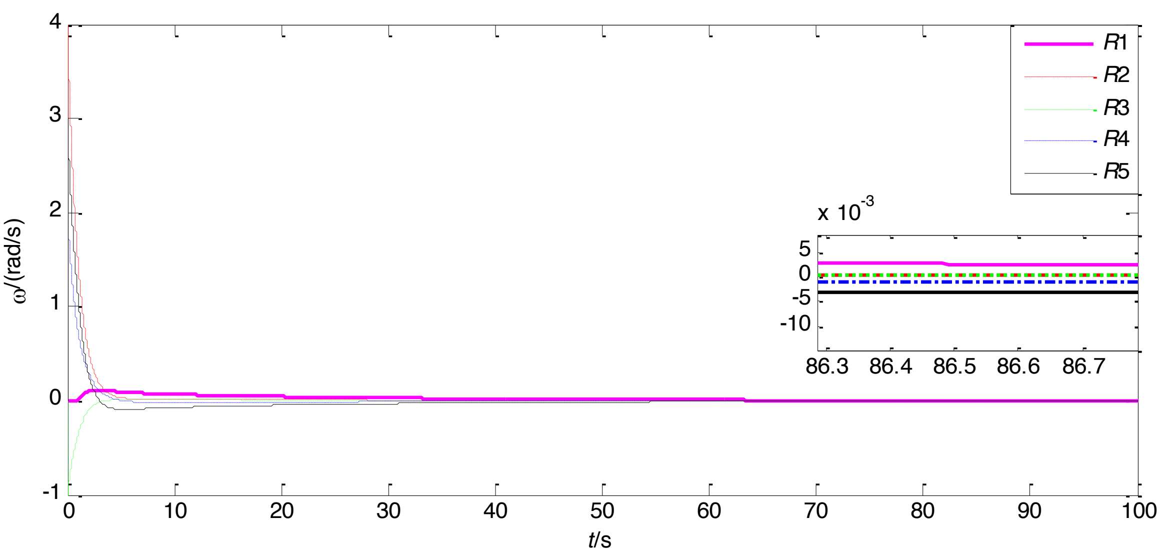


c

Figure 7. state changes of the 5 robots ($0<\tau_k\le 0.01\text{s}$, $\eta=10\%$, white noise)

a line velocity changes of the 5 robots ($0<\tau_k\le 0.01\text{s}$, $\eta=10\%$, white noise)

b azimuth changes of the 5 robots ($0<\tau_k\le 0.01\text{s}$, $\eta=10\%$, white noise)

c angular velocity changes of the 5 robots ($0<\tau_k\le 0.01\text{s}$, $\eta=10\%$, white noise)

It can be seen from Figure 6 that, when the MAS described as (35) is subjected to random short time delay ($0<\tau_k\le 0.01\text{s}$), packet loss rate ($1-\eta=10\%$) and environmental noise, under the effect of the controller designed in Theorem 1, the line velocities of agents with different initial values (1, 3, 2, 0, 5) tend to an identical value 0, and the line velocity convergence time of the MAS is about 11s; the azimuths of agents with different initial values (0, -2, 3, 1, 2) tend to an identical value 2.3, and the azimuth convergence time of the MAS is about 115s; and the angular velocities of agents with different initial values (0, 4, -1, 2, 3) tend to an identical value 0, and the angular velocity convergence time of the MAS is about 52s. As shown in Figure 7, when the MAS described as (35) is subjected to random short time delay ($0<\tau_k\le 0.01\text{s}$), packet loss rate

($1-\eta=90\%$) and environmental noise, under the effect of the controller designed in Theorem 1, the line velocities of agents with different initial values (1, 3, 2, 0, 5) tend to an identical value 0, and the line velocity convergence time of the MAS is about 12s; the azimuths of agents with different initial values (0, -2, 3, 1, 2) tend to an identical value 2.3, and the azimuth convergence time of the MAS is about 123s; and the angular velocities of agents with different initial values (0, 4, -1, 2, 3) tend to an identical value 0, and the angular velocity convergence time of the MAS is about 55s.

It can be concluded from Figure 6 and Figure 7 that, the data packet loss rate is bigger, i.e. the effective data transmission rate is smaller, the convergence time of the MAS will be longer, namely the time required for system to achieve consensus of $1-\eta=90\%$ is longer than the situation of $1-\eta=10\%$. The smaller figures in Figure 6 and Figure 7 are the zoom-in parts of state curves, and indicate that when there are uncertainties, the system can not achieve exactly the same state, but only reaches the convergence region, which is determined by the nature of H_∞ control.

It can be seen from Figure 6 and Figure 7 that, in the case A, i.e. when the MAS is subjected to random short time delay, packet loss and environmental noise, under the effect of the controller designed in Theorem 1, the MAS modeled as the asynchronous dynamical system can be stable, and all the subsystems can converge to the identical state, which means that the MAS reaches a consensus. So the consensus algorithm in case A is verified.

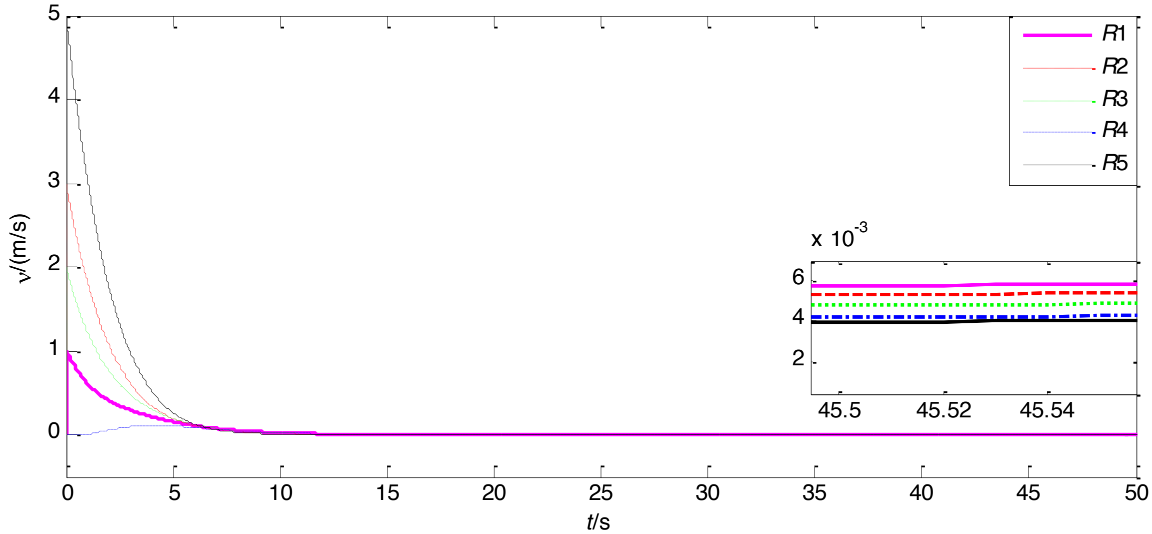
The consensus algorithm verification in case B

When the communication constraints including random long time delay, packet loss and environmental noise exist in the MAS, the simulation is also conducted for the running of five robots with the fixed communication topology shown as Figure 4. Namely, the consensus algorithm verification of MAS in case B is developed, i.e. the effectiveness of Theorem 2 is to be verified.

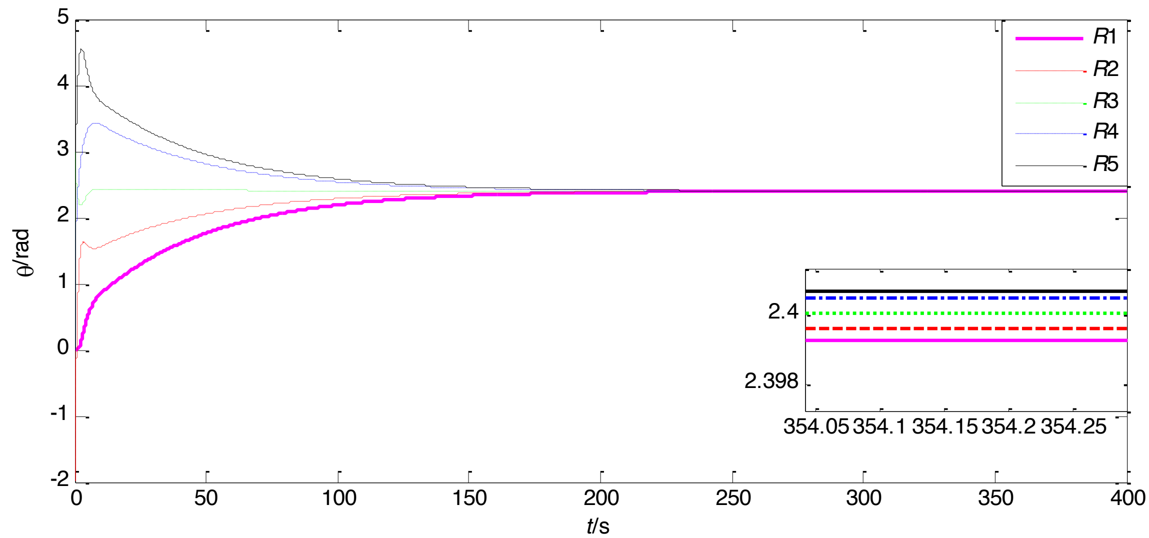
Suppose random long time delay is $0.01\text{s}<\tau_k<0.02\text{s}$, the discretization parameters of (35) are the same as the ones of random short delay. It is independent of the sizes of delay. The values of γ, α_1 and α_2 satisfying (25) remain the same. Suppose the successful transmission rate of data is $\eta=90\%$ ($0<\eta\le 1$), the LMIs (26) and (27) can be solved using the Matlab LMI toolbox. The convergence rate is $\alpha=1.2785>1$, and the controller gain matrix can be obtained from Theorem 2 and is shown below

$$\boldsymbol{K}=\begin{bmatrix} 0.0429 & 0.0025 & 0.0369 & 0.5186 & 0.0840 & 0.0978 & -0.0984 \\ -0.0001 & -0.0018 & -0.0173 & 0.0340 & 0.5720 & -0.0480 & 0.0474 \end{bmatrix}.$$

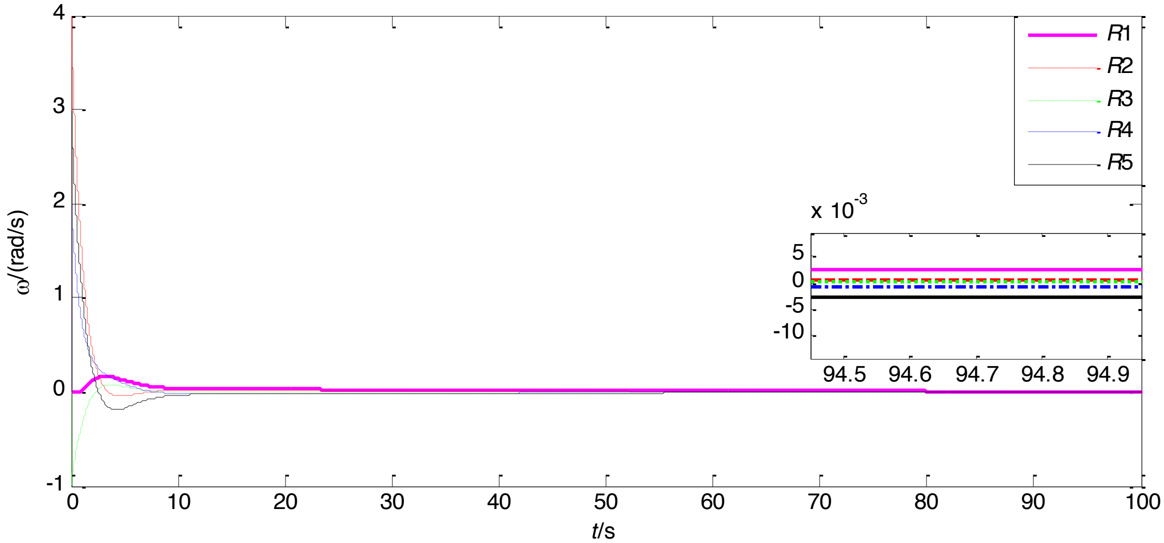
When random long delay ($0.01\text{s}<\tau_k<0.02\text{s}$), packet loss ($1-\eta=10\%$) and environmental noise shown in Figure 5 exist in the MAS described in (35), under the effect of the γ-suboptimal H_∞ controller obtained by solving LMIs in Theorem 2, the corresponding simulation results are shown in Figure 8. When packet loss rate is ($1-\eta=90\%$) and other simulation conditions remain unchanged, the corresponding simulation results are given in Figure 9.


R1
R2
R3
R4
R5
x 10^-3
6
4
2
45.5
45.52
45.54
5
4
3
2
1
0
v/(m/s)
0
5
10
15
20
25
30
35
40
45
50
t/s

a


R1
R2
R3
R4
R5
2.4
2.398
354.05
354.1
354.15
354.2
354.25
5
4
3
2
1
0
-1
-2
θ/rad
0
50
100
150
200
250
300
350
400
t/s

b



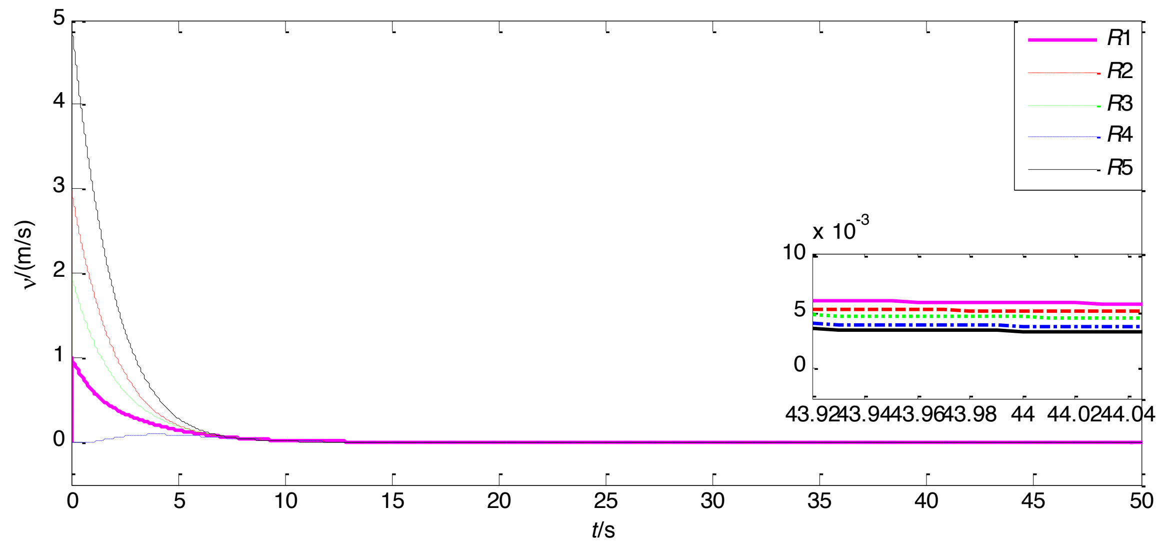
c

Figure 8. state changes of the 5 robots ($0.01\text{s} < \tau_k < 0.02\text{s}$, $\eta = 90\%$, white noise)

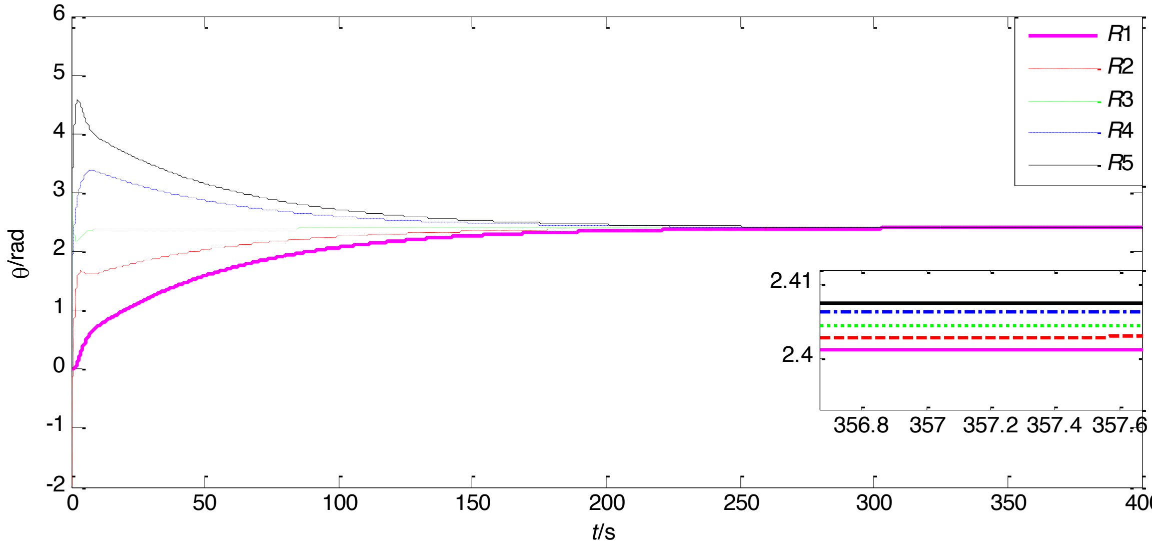
a line velocity changes of the 5 robots ($0.01\text{s} < \tau_k < 0.02\text{s}$, $\eta = 90\%$, white noise)

b azimuth changes of the 5 robots ($0.01\text{s} < \tau_k < 0.02\text{s}$, $\eta = 90\%$, white noise)

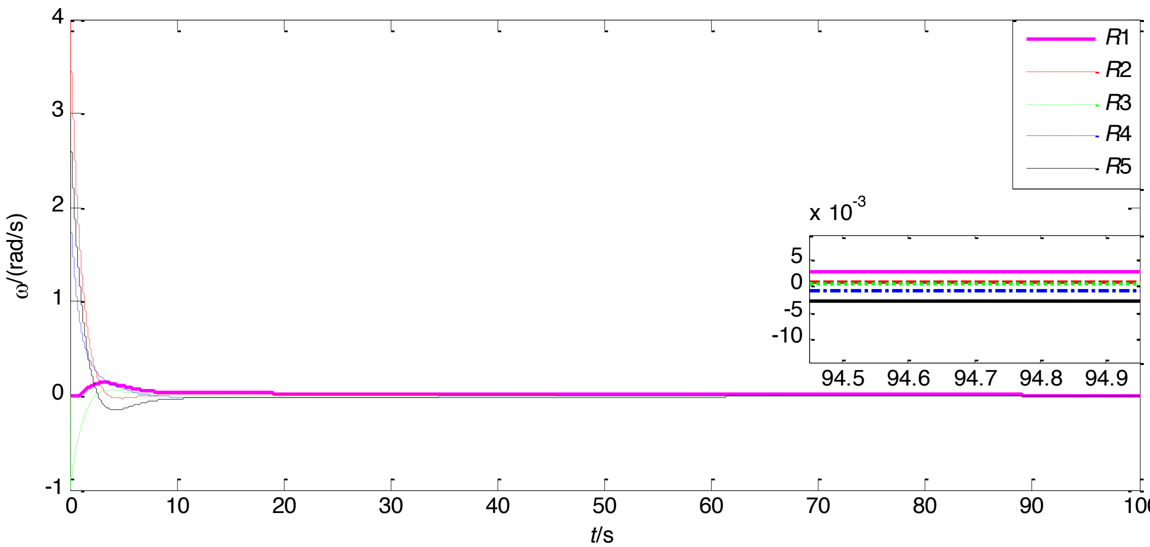
c angular velocity changes of the 5 robots ($0.01\text{s} < \tau_k < 0.02\text{s}$, $\eta = 90\%$, white noise)



a



b



c

Figure 9. state changes of the 5 robots ($0.01\text{s} < \tau_k < 0.02\text{s}$, $\eta = 10\%$, white noise)

a line velocity changes of the 5 robots ($0.01\text{s} < \tau_k < 0.02\text{s}$, $\eta = 10\%$, white noise)

b azimuth changes of the 5 robots ($0.01\text{s} < \tau_k < 0.02\text{s}$, $\eta = 10\%$, white noise)

c angular velocity changes of the 5 robots ($0.01\text{s} < \tau_k < 0.02\text{s}$, $\eta = 10\%$, white noise)

It can be seen from Figure 8 that, when the MAS described as (35) is subjected to random long time delay ($0.01\text{s} < \tau_k < 0.02\text{s}$), packet loss rate ($1-\eta = 10\%$) and environmental noise, under the effect of the controller designed in Theorem 2, the line velocities of agents with different initial values (1, 3, 2, 0, 5) tend to an identical value 0, and the line velocity convergence time of the MAS is about 12s; the azimuths of agents with different initial values (0, -2, 3, 1, 2) tend to an identical value 2.3, and the azimuth convergence time of the MAS is about 230s; and the angular velocities of agents with different initial values (0, 4, -1, 2, 3) tend to an identical value 0, and the angular velocity convergence time of the MAS is about 58s. As shown in Figure 9, when the MAS described as (35) is subjected to random short time delay ($0.01\text{s} < \tau_k < 0.02\text{s}$), packet loss rate ($1-\eta=90\%$) and environmental noise, under the effect of the controller designed in Theorem 2, the line velocities of agents with different initial values (1, 3, 2, 0, 5) tend to an identical value 0, and the line velocity convergence time of the MAS is about 13s; the azimuths of agents with different initial values (0, -2, 3, 1, 2) tend to an identical value 2.3, and the azimuth convergence time of the MAS is about 260s; and the angular velocities of agents with different initial values (0, 4, -1, 2, 3) tend to an identical value 0, and the angular velocity convergence time of the MAS is about 62s.

It can be concluded from Figure 6 and Figure 8 that, the time delay is longer, and the convergence time of the MAS will be longer, namely the time required for system to achieve consensus of $0.01\text{s} < \tau_k < 0.02\text{s}$ is longer than the situation of $0 < \tau_k \leq 0.01\text{s}$. It can also be concluded from Figure 8 and Figure 9 that, the data packet loss rate is bigger, i.e. the effective data transmission rate is smaller, the convergence time of the MAS will be longer, namely the time required for system to achieve consensus of $1-\eta=90\%$ is longer than the situation of $1-\eta=10\%$. Similarly, the smaller figures in Figure 8 and Figure 9 are the zoom-in parts of state curves. Due to the nature of H_∞ control, the system only reaches the convergence region.

It can be seen from Figure 8 and Figure 9 that, in the case B, i.e. when the MAS is subjected to random long time delay, packet loss and environmental noise, under the effect of the controller designed in Theorem 2, the MAS modeled as the asynchronous dynamical system can be stable, and all the subsystems can converge to the identical state, which means that the MAS reaches a consensus. So the consensus algorithm in case B is verified.

Specially note, the environmental noise is taken into account in system analysis and the entire simulation process. As can be seen in Figures 6-9, the system is still stable under the influence of environmental noise, so the MAS is stable with strong robustness. Comparing the effects of different time delays and packet loss probabilities on the consensus of MAS, the longer time is required for the system to achieve consensus if either longer time delay or bigger packet loss probability occurs.

Conclusion

The consensus for MAS with directed information flow and fixed communication topology was analyzed by taking several mathematical tools from algebraic graph theory, matrix theory, and H_∞ control theory.

Firstly, for the convenience of analysis, the MAS with communication constraints was divided into two cases: (A) the system is subjected to the random short time delay, packet loss and environmental noise; (B) the system is subjected to the random long time delay, packet loss and environmental noise. The MAS with communication constraints can be transformed into an asynchronous dynamical system; thus, the consensus control of MAS was equivalent to the robust H_∞ control of an asynchronous dynamical system. Secondly, the sufficient conditions for robust exponential stability of MAS were given and the corresponding γ-suboptimal H_∞ controllers were designed. Thirdly, the admissible packet loss probability guaranteeing the robust exponential stability of the system was also obtained. Finally, the corresponding simulation results carried out through Matlab TrueTime toolbox were given to demonstrate the effectiveness of the algorithms.

Since the agents as nodes of the MAS are moving, it is not hard to imagine that some of the existing communication links can fail due to the existence of an obstacle between two agents. The situation can arise where new links between nearby agents are created because the agents come to an effective range of detection with respect to each other. In terms of the network communication topology, this means that certain number of edges are added or removed from the graph. There are some problems to be solved in the future, such as consensus in MAS with communication constraints under switching communication topology.

Conflict of interest

The authors declare that there is no conflict of interest.

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